X : Hadamard space

$f : X \to \mathbb{R}$ (lower semi-) continuous & convex

$$J_\lambda^f(x) := \underset{z \in X}{\operatorname{argmin}}\ f(z) + \frac{1}{2\lambda} d(x, z)^2$$

↑ strongly convex

a minimizer uniquely exist.

__Lem__ $\quad x' := J_\lambda^f(x)$

$$d(x', y)^2 \leq d(x, y)^2 - 2\lambda\left(f(x') - f(y)\right)$$

$(\forall y \in X)$

$(y \to x)$ $\quad f(x') + \frac{1}{2\lambda} d(x', x)^2 \leq f(x)$

$$f(x') + \frac{1}{2\lambda} d(x, x') \leq f((1-t)x' + ty) + \frac{1}{2\lambda} d(x, (1-t)x' + ty)^2$$

(f : convex & $d^2(x, \cdot)$: strongly convex)

$$\leq (1-t) f(x') + t f(y) + \frac{1-t}{2\lambda} d(x, x')^2 + \frac{t}{2\lambda} d(x, y)^2 - \frac{t(1-t)}{2\lambda} d(x', y)^2$$

$$\Rightarrow \quad 0 \leq -t f(x') + t f(y) - \frac{t}{2\lambda} d(x,x')^2 + \frac{t}{2\lambda} d(x,y)^2 - \frac{t(1-t)}{2\lambda} d(x',y)^2$$

divide by t, $t \to 0$

$$\Rightarrow \quad 0 \leq -f(x') + f(y) - \frac{1}{2\lambda} d(x,x') + \frac{1}{2\lambda} d(x,y)^2 - \frac{1}{2\lambda} d(x',y)^2$$

$$\Rightarrow \quad d(x',y)^2 \leq d(x,y)^2 - d(x,x')^2 - 2\lambda\big(f(x') - f(y)\big)$$

$$\leq d(x,y)^2 - 2\lambda\big(f(x') - f(y)\big)$$

$\blacksquare$

Convergence of SPPA

Claim (Bačák)

$$d(x_{kN+N}, y)^2 \leq d(x_{kN}, y)^2 - 2\lambda_k\left(f(x_{kN}) - f(y)\right) + 2\lambda_k^2 L^2 N(N+1)$$

By above lemma

$$d(x_{kN+i}, y)^2 \leq d(x_{kN+i-1}, y)^2 - 2\lambda_k\left(f_i(x_{kN+i}) - f_i(y)\right)$$
$$(\forall y \in X,\ i = 1, 2, \cdots, N)$$

Summing up for $i = 1, 2, \cdots N$

$$d(x_{kN+N}, y)^2 \leq d(x_{kN}, y)^2 - 2\lambda_k \sum_{i=1}^{N}\left(f_i(x_{kN+i}) - f_i(y)\right)$$

$$= d(x_{kN}, y)^2 - 2\lambda_k\left(f(x_{kN}) - f(y)\right) + 2\lambda_k \sum_{i=1}^{N}\left(f_i(x_{kN}) - f_i(x_{kN+i})\right)$$

$$f_i(x_{kN}) - f_i(x_{kN+i}) \leq L\, d(x_{kN}, x_{kN+i})$$

$$\leq L\left(d(x_{kN}, x_{kN+1}) + \cdots + d(x_{kN+i-1}, x_{kN+i})\right)$$

By lemma,

$$f_i(x_{kN+i}) + \frac{1}{2\lambda_k} d(x_{kN+i-1}, x_{kN+i})^2 \leq f_i(x_{kN+i-1})$$

$$\to \quad d(x_{kN+i-1}, x_{kN+i}) \leq 2\lambda_k \frac{f_i(x_{kN+i-1}) - f_i(x_{kN+i})}{d(x_{kN+i-1}, x_{kN+i})}$$

$$\leq 2\lambda_k L$$

$$\to \quad d(x_{kN+N}, y)^2 \leq d(x_{kN}, y)^2 - 2\lambda_k\left(f(x_{kN}) - f(y)\right) + 2\lambda_k L \underbrace{\sum_{i=1}^{N} 2\lambda_k i L}_{2\lambda_k^2 L^2 N(N+1)} \qquad \blacksquare$$

Now f is ε-strongly convex for $\varepsilon > 0$

Let $x^* :=$ the unique minimizer of f.

$$\Longrightarrow \quad \frac{\varepsilon}{2} d(x, x^*)^2 \leq f(x) - f(x^*) \qquad (x \in X)$$

$$(1-t) f(x) + t f(x^*) \geq f((1-t)x + t x^*) + \frac{\varepsilon t(1-t)}{2} d(x, x^*)^2$$

$$\Rightarrow \quad f(x) \geq f(x^*) + \underbrace{\frac{f((1-t)x + t x^*) - f(x^*)}{1-t}}_{\geq 0} + \frac{\varepsilon t}{2} d(x, x^*)^2$$

$$t \to 1$$

$$\boxed{d(x_{(k+1)N}, x^*)^2 \leq (1 - \lambda_k \varepsilon)\, d(x_{kN}, x^*)^2 + 2\lambda_k^2 L^2 N(N+1)}$$

$$\lambda_k := \frac{1}{\varepsilon} \frac{1}{k+1}, \quad \beta := \frac{2L^2 N(N+1)}{\varepsilon^2}$$

$$a_k := d(x_{kN}, x^*)^2$$

Lem Let $a_k \geq 0$ be a sequence s.t.

$$a_{k+1} \leq \left(1 - \frac{1}{k+1}\right) a_k + \frac{\beta}{(k+1)^2}$$

for $\beta > 0$

$$\Rightarrow a_k \leq \beta \frac{1 + \log(k+1)}{k+1}$$

$k=0$

$$\beta \frac{1+\log 2}{2} \leq \beta$$

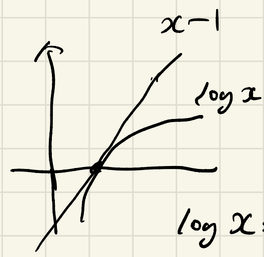
∵ induction; ok for $k=1$, suppose true for $k \geq 1$

$$a_{k+1} \leq \left(1 - \frac{1}{k+1}\right) a_k + \frac{\beta}{(k+1)^2}$$

$$\leq \frac{k}{k+1} \beta \frac{1 + \log k+1}{k+1} + \frac{\beta}{(k+1)^2}$$

$$= \beta \frac{k+1 + k \log k+1}{(k+1)^2}$$

$$= \frac{\beta}{k+2} \times \left\{ \frac{k+2}{k+1} + \frac{(k+2)k}{(k+1)^2} \left(\log(k+2) + \underbrace{\log \frac{k+1}{k+2}}_{\leq \frac{k+1}{k+2} - 1 = \frac{-1}{k+2}} \right) \right\}$$



$$\log x \leq x - 1$$

$$\leq \frac{\beta}{k+2}\left\{ \frac{(k+2)(k+1)+(k+2)k\log(k+2)-k}{(k+1)^2} \right\}$$

$$= \frac{\beta}{k+2} \frac{k^2+2k+2+(k+2)k\log(k+2)}{(k+1)^2}$$

$$= \frac{\beta}{k+2}\left(1+\frac{1+(k+2)k\log(k+2)}{(k+1)^2}\right)$$

$1 \leq \log k+2 \; (k \geq 1)$

$$\leq \frac{\beta}{k+2}\left(1+\frac{\log k+2+(k+2)k\log(k+2)}{(k+1)^2}\right)$$

$$= \frac{\beta}{k+2}\left(1+\log(k+2)\times\frac{\cancel{1+k^2+2k}}{\cancel{(k+1)^2}}\right)$$

$$\Rightarrow a_{k+1} \leq \beta\frac{1+\log k+2}{k+2}$$

$\blacksquare$