# Geometric Complexity Theory: No Occurrence Obstructions for Determinant vs Permanent

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# Problem and Main Result

# Permanent versus determinant

- How many arithmetic operations are sufficient to evaluate the permanent of an $m$ by $m$ matrix $(x_{ij})$?

$$\mathrm{per}_m := \sum_{\pi \in S_m} x_{1\pi(1)} \cdots x_{m\pi(m)}$$

- Best known algorithm: $O(m2^m)$ operations
- The determinant $\det_n$ can be evaluated with $\mathrm{poly}(n)$ operations

$$\det_n := \sum_{\pi \in S_n} sgn(\pi)\, x_{1\pi(1)} \cdots x_{n\pi(n)}$$

- Work over $\mathbb{C}$

# Valiant's Conjecture

- Are there linear forms $a_{ij} = a_{ij}(x, z)$ in $x_{ij}$ and $z$ such that $(n \geq m)$

$$z^{n-m}\mathrm{per}_m = \det \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \qquad ? \qquad (*)$$

- Impossible for $n = m > 2$ (Polya)
- Possible for $n \leq 2^m - 1$ (Valiant, Grenet)
- $n \geq \frac{1}{2}m^2$ (Mignon & Ressayre 2004)

- Valiant's Conjecture (1979): (*) impossible for $n = poly(m)$

- Conjecture equivalent to the separation $\mathrm{VBP} \neq \mathrm{VNP}$ of complexity classes
- $\mathrm{P} \neq \mathrm{NP}$ nonuniformly implies $\mathrm{VBP} \neq \mathrm{VNP}$ under GRH (B, 2000)

# Orbit closure of $\det_n$

- Approach by Mulmuley and Sohoni (2001) based on algebraic geometry and representation theory
- Idea of orbit closures already in Strassen (1987) for tensor rank
- $n$th symmetric power $\mathrm{Sym}^n V^*$ of dual space $V^*$ with natural action of group $G := \mathrm{GL}(V)$
- Orbit $G \cdot f := \{g \cdot f \mid g \in G\}$ of $f \in \mathrm{Sym}^n V^*$
- Take $V := \mathbb{C}^{n \times n}$, $N = n^2$, view $\det_n$ as element of $\mathrm{Sym}^n V^*$
- Orbit closure w.r.t. Euclidean or Zariski topology

$$\Omega_n := \overline{\mathrm{GL}_{n^2} \cdot \det_n} \subseteq \mathrm{Sym}^n(\mathbb{C}^{n \times n})^*$$

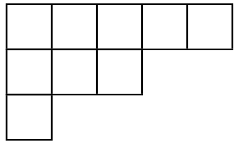
- $\Omega_2 = \mathrm{Sym}^2(\mathbb{C}^{2\times 2})^*$; $\Omega_3$ known (Hüttenhain & Lairez '16); $\Omega_4$ already unknown

# Orbit Closure Conjecture

- Padded permanent $X_{11}^{n-m}\mathrm{per}_m \in \mathrm{Sym}^n(\mathbb{C}^{n\times n})^*$, where $n > m$
- **Orbit Closure Conjecture (M-S 2001)**

  For all $c \in \mathbb{N}_{\geq 1}$ we have $X_{11}^{m^c-m}\mathrm{per}_m \notin \Omega_{m^c}$ for infinitely many $m$.
- The Orbit Closure Conjecture implies Valiant's Conjecture

# Splitting into irreps

- Action of group $G = \mathrm{GL}(V)$ on $\mathrm{Sym}^n V^*$ induces action on its graded coordinate ring $\mathbb{C}[\mathrm{Sym}^n V^*] = \oplus_{d \in \mathbb{N}} \mathrm{Sym}^d \mathrm{Sym}^n V$
- The plethysms $\mathrm{Sym}^d \mathrm{Sym}^n V$ splits into irreducible $G$-representations $\mathcal{W}_\lambda$ (Weyl modules), labeled by partitions $\lambda \vdash dn$ into at most $\dim V = n^2$ parts
- Visualize partition as Young diagram: $(5,3,1) \vdash 9$ write as 
- Size $|(5,3,1)| := 9$ is number of boxes; length $\ell(5,3,1) = 3$ is number of parts
- $\mathbb{C}[\Omega_n]$ denotes coordinate ring of $\Omega_n$
- Restriction of polynomial maps to $\Omega_n$ gives surjective $G$-equivariant linear map:

$$\mathrm{Sym}^d \mathrm{Sym}^n V = \mathbb{C}[\mathrm{Sym}^n V^*] \twoheadrightarrow \mathbb{C}[\Omega_n]_d$$

- Say $\lambda$ occurs in $\mathbb{C}[\Omega_n]_d$ if it contains a copy of $\mathcal{W}_\lambda$

# Obstructions

- $Z_{n,m}$ denotes orbit closure of the padded permanent ($n > m$):

$$Z_{n,m} := \overline{\mathrm{GL}_{n^2} \cdot X_{11}^{n-m}\mathrm{per}_m} \subseteq \mathrm{Sym}^n(\mathbb{C}^{n\times n})^*. \tag{1}$$

- Suppose $X_{11}^{n-m}\mathrm{per}_m \in \Omega_n$
- Then $Z_{n,m} \subseteq \Omega_n$ and restriction gives $\mathbb{C}[\Omega_n] \twoheadrightarrow \mathbb{C}[Z_{n,m}]$
- Schur's lemma: if $\lambda$ occurs in $\mathbb{C}[Z_{n,m}]$, then $\lambda$ occurs in $\mathbb{C}[\Omega_n]$
- Partition $\lambda$ violating this condition is called occurrence obstruction.
- Its existence would prove $Z_{n,m} \not\subseteq \Omega_n$
- Schur's lemma also gives inequality of multiplicities:

$$\mathrm{mult}_\lambda \mathbb{C}[\Omega_n] \;\geq\; \mathrm{mult}_\lambda \mathbb{C}[Z_{n,m}]$$

- Partition $\lambda$ violating this inequality is called multiplicity obstruction.

  See Dörfler, Ikenmeyer, Panova 2019

# Main Result

M-S suggested the following conjecture

**Occurrence Obstruction Conjecture (M-S 2001)**

For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many $m$, there exists a partition $\lambda$ occurring in $\mathbb{C}[Z_{m^c,m}]$ but not in $\mathbb{C}[\Omega_{m^c}]$.

Occurrence Obstruction Conjecture implies Orbit Closure Conjecture

Unfortunately, the Occurrence Obstruction Conjecture is false!

**Thm. (B, Ikenmeyer, Panova, FOCS 16, J. AMS '18)**

Let $n, d, m$ be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If $\lambda$ occurs in $\mathbb{C}[Z_{n,m}]$, then $\lambda$ also occurs in $\mathbb{C}[\Omega_n]$. In particular, the Occurrence Obstruction Conjecture is false.

Before this, [IP16] (Ikenmeyer, Panova FOCS 16) had a similar result showing that the Orbit Closure Conjecture cannot be resolved via Kronecker coefficients

# No occurrence obstructions for Waring rank

- Waring rank (symmetric tensor rank) of $p \in \mathrm{Sym}^n V^*$: minimum $r$ s.t. $p = \varphi_1^n + \ldots + \varphi_r^n$ for linear forms $\varphi_i \in V^*$
- Can prove exponential lower bound on Waring rank of $\mathrm{det}_n$, $\mathrm{per}_n$
- May think of proving lower bounds on Waring rank by studying orbit closure

$$\mathrm{PS}_n := \overline{\mathrm{GL}_{n^2} \cdot (X_1^n + \cdots + X_{n^2}^n)} \subseteq \mathrm{Sym}^n(\mathbb{C}^{n^2})^*.$$

**Corollary**

Let $n, d, m$ be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If $\lambda$ occurs in $\mathbb{C}[Z_{n,m}]$, then $\lambda$ also occurs in $\mathbb{C}[\mathrm{PS}_n]$. Moreover, the permanent can be replaced by any homogeneous polynomial $p$ of degree $m$ in $m^2$ variables.

Hence strategy of occurrence obstructions cannot even be used in weak model of $\mathrm{PS}_n$ against padded polynomials!

# Outline and Ingredients of Proof

# Kadish & Landsberg's observation

- body $\bar{\lambda}$ of $\lambda$: obtained by removing the first row of $\lambda$,

**Kadish & Landsberg '14**

If $\lambda \vdash nd$ occurs in $\mathbb{C}[Z_{n,m}]_d$, then $\ell(\lambda) \leq m^2$ and $|\bar{\lambda}| \leq md$.

- $|\bar{\lambda}| \leq md$ is equivalent to $\lambda_1 \geq (n-m)d$: $\lambda$ must have a very long first row if $n$ is substantially larger than $m$
- This is the only information we exploit about the orbit closure $Z_{n,m}$ of the padded permanent
- Can replace the permanent by any homogeneous polynomial $p$ of degree $m$ in $m^2$ variables
- Kadish & Landsberg also crucially used in [IP16]

# Semigroup property

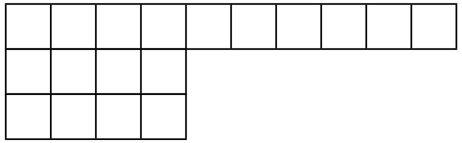
- Need to show that many partitions $\lambda$ occur in $\mathbb{C}[\Omega_n]$
- For this establish the occurrence of certain basic shapes in $\mathbb{C}[\Omega_n]$
- Then get more shapes by

**Semigroup Property**

If $\lambda$ occurs in $\mathbb{C}[\Omega_n]$ and $\mu$ occurs in $\mathbb{C}[\Omega_n]$, then $\lambda + \mu$ occurs in $\mathbb{C}[\Omega_n]$.

- Pf. There are highest weight vectors $F_\lambda, F_\mu \in \mathbb{C}[\Omega_n]$ of weight $\lambda, \mu$, resp. The product $F_\lambda \cdot F_\mu$ is a highest weight vectors of weight $\lambda + \mu$. $\square$
- Semigroup Property also crucially used in [IP16]

# Basic building blocks

- Denote by $(k \times \ell)^{\sharp nk}$ the rectangular diagram $k \times \ell$ with $k$ rows of length $\ell$, to which a row has been appended s.t. we get $nk$ boxes
- $(3 \times 4)^{\sharp 18} =$ 

  - **Prop. RER (Row Extended Rectangles)**

    Let $n \geq k\ell$ and $\ell$ be even. Then $(k \times \ell)^{\sharp nk}$ occurs in $\mathbb{C}[\Omega_n]_k$.

- The only property of $\Omega_n$ used in the proof is that $\Omega_n$ contains many padded power sums (follows from universality of determinant)

**Prop. PPS (Padded Power Sums)**

- Let $X, \varphi_1, \ldots, \varphi_k$ be linear forms on $\mathbb{C}^{n \times n}$ and assume $n \geq sk$. Then the power sum $X^{n-s}(\varphi_1^s + \cdots + \varphi_k^s)$ of $k$ terms of degree $s$, padded to degree $n$, is contained in $\Omega_n$.

# Strategy of proof of main result

- Suppose have even $\lambda \vdash nd$ such that $n \geq m^{25}$ and $\lambda$ occurs in $\mathbb{C}[Z_{n,m}]$. Want to show that $\lambda$ occurs in $\mathbb{C}[\Omega_n]$.
- By [KL14] we have $\ell(\lambda) \leq m^2$ and $|\bar{\lambda}| \leq md$.
- Distinguish two cases
- CASE 1: If the degree $d$ is large (say $d \geq 24m^6$), we proceed as in [IP16]: we decompose body $\bar{\lambda}$ into a sum of even rectangles
- Since $n$ and $d$ are sufficiently large in comparison with $m$, can write (!) $\lambda$ as a sum of row extended rectangles $(k \times \ell)^{\sharp nk}$, where $n \geq k\ell$.
- By Prop. RER the row extended rectangles occur in $\mathbb{C}[\Omega_n]$. The semigroup property implies that $\lambda$ occurs in $\mathbb{C}[\Omega_n]$.

# Case of small degree

- CASE 2: If the degree $d$ is small, we rely on the following crucial result. Recall $V = \mathbb{C}^{n \times n}$.

**Prop. ALL**

Let $\lambda \vdash nd$ be such that $|\bar{\lambda}| \leq md$ and $md^2 \leq n$ for some $m$.

Then every highest weight vector of weight $\lambda$ in $\mathrm{Sym}^d \mathrm{Sym}^n V$, viewed as a degree $d$ polynomial function on $\mathrm{Sym}^n V^*$, does not vanish on $\Omega_n$.

In particular, if $\lambda$ occurs in $\mathrm{Sym}^d \mathrm{Sym}^n V$, then $\lambda$ occurs in $\mathbb{C}[\Omega_n]_d$.

- The proof relies on new insights on "lifting highest weight vectors" in plethysms
- This is related to known stability property of plethysms, for which we obtain new proofs
- For treating noneven partitions, need more bulding blocks (row and column extended rectangles) and more tricks

# Some Basics

# Polynomials as symmetric tensors

- $d$th tensor power tensor $\bigotimes^d V$ of $V \simeq \mathbb{C}^N$
- Symmetrizing projection

$$\bigotimes^d V \twoheadrightarrow \mathrm{Sym}^d V,\ w \mapsto \frac{1}{d!} \sum_{\pi \in \mathfrak{S}_d} \pi(w)$$

- Have pairing (contraction)

$$\mathrm{Sym}^d V^* \times \mathrm{Sym}^d V, (F, w) \mapsto \langle F, w \rangle$$

- Polarization: view homogeneous polynomials as symmetric tensors

$$\mathrm{Sym}^d V \xrightarrow{\sim} \mathbb{C}[V^*]_d,\ F \mapsto \left(v \mapsto \langle F, v^{\otimes d} \rangle\right) := F(v)$$

- In particular, $\mathrm{Sym}^d \mathrm{Sym}^n V \xrightarrow{\sim} \mathbb{C}[\mathrm{Sym}^n V^*]_d$, and $F(p) := \langle F, p^{\otimes d} \rangle$ for $p \in \mathrm{Sym}^n V^*$.

# Highest weight vectors

- Suppose that $\mathscr{V}$ is rational $\mathrm{GL}_N(\mathbb{C})$-module
- $U_N \subseteq \mathrm{GL}_N(\mathbb{C})$: subgroup of upper triangular matrices with 1's on main diagonal
- $f \in \mathscr{V}$ is called highest weight vector of weight $\lambda \in \mathbb{Z}^N$ iff
  - $f \neq 0$ is $U_N$-invariant, i.e., $u \cdot f = f$ for all $u \in U_N$,
  - $f$ is a weight vector of weight $\lambda$, i.e., for all $\alpha_i \in \mathbb{C}^\times$ $\mathrm{diag}(t_1, \dots, t_N) \cdot f = t_1^{\lambda_1} \cdots t_N^{\lambda_N} f$
- Have $\lambda_1 \geq \ldots \geq \lambda_N$, so $\lambda$ is partition if its entries are nonnegative.
- $\mathsf{HWV}_\lambda(\mathscr{V})$: vector space of highest weight vectors of weight $\lambda$.
- Ex. $\mathsf{HWV}_\lambda(\mathrm{Sym}^d \mathbb{C}^N) = \mathbb{C} e_1^{\otimes d}$
- Known:

  $$\mathscr{V} \text{ irreducible} \iff \dim \mathsf{HWV}_\lambda(\mathscr{V}) = 1$$

  $\lambda$ determines isomorphy type of $\mathscr{V}$; call $\mathscr{V}$ of type $\lambda$
- Known: if $\lambda$ is rectangular, $\lambda_1 = \ldots = \lambda_N$, then $f$ is $\mathrm{SL}_N$-invariant

# Constructing HWVs in tensor powers

- How to construct highest weight vectors in $\bigotimes^d V$?
- $v_{j\times 1} := e_1 \wedge e_2 \wedge \cdots \wedge e_j$ is highest weight vector of weight $j \times 1$.
- Let $\lambda \vdash D$ and $\mu$ denote transpose of $\lambda$, so $\mu_i$ denotes number of boxes in $i$-th column of $\lambda$.
- Then

$$v_\lambda := v_{\mu_1\times 1} \otimes \ldots \otimes v_{\mu_{\lambda_1}\times 1} \ \in \ \bigotimes^D V$$

  is highest weight vector of weight $\lambda$
- Schur-Weyl duality implies

$$\mathrm{span}\{\pi v_\lambda \mid \pi \in \mathfrak{S}\} = \mathsf{HWV}_\lambda(\bigotimes^D V)$$

# Constructing HWVs in plethysms

- Consider block decomposition of positions for $D = 6$:

$$\{1,2,3\} \cup \{4,5,6\}$$

- Symmetrize within blocks and simultaneously permute blocks

$$\Sigma_{2,3} : \, : \bigotimes^6 V \to \mathrm{Sym}^2 \mathrm{Sym}^3 V$$

- For $\lambda \vdash 6$ we obtain

$$\mathrm{span}\{\Sigma_{2,3} \pi v_\lambda \mid \pi \in \mathfrak{S}_6\} = \mathsf{HWV}_\lambda(\mathrm{Sym}^2 \mathrm{Sym}^3 V)$$

- Think of $\Sigma_{2,3} v_\lambda$ as encoded by

$$\pi \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} \mapsto \pi \begin{array}{|c|c|c|c|} \hline a & a & b & b \\ \hline a & b \\ \cline{1-2} \end{array}.$$

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$$(14) \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} = \begin{array}{|c|c|c|c|} \hline 4 & 3 & 5 & 6 \\ \hline 2 & 1 \\ \cline{1-2} \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline b & a & b & b \\ \hline a & a \\ \cline{1-2} \end{array}.$$

- Encode HWVs of $\mathrm{Sym}^d \mathrm{Sym}^n V$ by Young tableau $T$ of shape $\lambda \vdash dn$, filled with $d$ different letters, where each letter occurs $n$ times

# Highest weight vectors as polynomial functions

- How to show that $\lambda$ occurs in $\mathbb{C}[\Omega_n]$?
- Recall: $F \in \mathrm{Sym}^d\mathrm{Sym}^n\mathbb{C}^N$ is highest weight vector of weight $\lambda$ if

$$\begin{pmatrix} t_1 & * & * & * \\ & t_2 & * & * \\ & & \ddots & \vdots \\ & & & t_N \end{pmatrix} \cdot F = t_1^{\lambda_1} \cdots t_N^{\lambda_N} F \qquad \text{for all } t_i \in \mathbb{C}^*$$

- View $F$ as homogeneous degree $d$ polynomial function

$$F \colon \mathrm{Sym}^n(\mathbb{C}^N)^* \to \mathbb{C}, \quad F(p) = \langle F, p^n \rangle$$

- Restriction of polynomial functions is surjective and $\mathrm{GL}_N$-equivariant

$$\mathrm{Sym}^d\mathrm{Sym}^n\mathbb{C}^N \simeq \mathbb{C}[\mathrm{Sym}^n(\mathbb{C}^N)^*] \twoheadrightarrow \mathbb{C}[\overline{\mathrm{GL}_N \cdot p}]$$

- Essential observation:

If $F(p) \neq 0$, then $\lambda$ occurs in $\mathbb{C}[\overline{\mathrm{GL}_N \cdot p}]$

# Fundamental Invariants

# Fundamental invariants

- Suppose $n$ is even. Howe ('87) showed:
- If $d < N$, then $\mathrm{Sym}^d\mathrm{Sym}^n\mathbb{C}^N$ doesn't have a nonzero $\mathrm{SL}_N$-invariant
- If $d = N$, then $\mathrm{Sym}^d\mathrm{Sym}^n\mathbb{C}^N$ has exactly one $\mathrm{SL}_N$-invariant $F_{n,N}$, up to scaling, the fundamental invariant, already known to Cayley as a "hyperdeterminant"
- View $F_{n,N}$ as a homogeneous degree $N$ polynomial map

$$F_{n,N} \colon \mathrm{Sym}^n(\mathbb{C}^N)^* \to \mathbb{C}$$

- For $p = \sum_{1 \le j_1,\ldots,j_n \le N} v(j_1,\ldots,j_n) X_{j_1} \cdots X_{j_n}$ with symmetric coefficients

$$F_{n,N}(p) = \sum_{\sigma_1,\ldots,\sigma_n \in S_N} \mathrm{sgn}(\sigma_1)\cdots\mathrm{sgn}(\sigma_n) \prod_{i=1}^{N} v(\sigma_1(i),\ldots,\sigma_n(i))$$

- Ex. $n = 2$: $F_{2,N}(p) = N!\det(v)$ where $v$ is symmetric matrix
- For $g \in \mathrm{GL}_N$

$$F_{n,N}(g \cdot p) = \det(g)^n F_{n,N}(p)$$

# Evaluating fundamental invariants

- [B, Ikenmeyer '17]: systematic investigation of fundamental invariants
- $F_{n,N}$ is a highest weight vector (weight $N \times n$)
- It is not easy to prove $F_{n,N}(p) \neq 0$
- Seemingly simple example ($n$ even)

$$F_{n,n}(X_1 \cdots X_n) = \frac{1}{n!}(\#\{\text{col. even latin squares}\} - \#\{\text{col. odd latin squares}\}) \stackrel{?}{=} 0$$

- This is unknown: Alon-Tarsi Conjecture!
- Essential for basic building blocks: prove $F_{n,N}(X_1^n + \ldots + X_N^n) \neq 0$ by writing it as sum of squares [B, Christandl, Ikenmeyer '11]

# Lifting of Highest Weight Vectors

# Lifiting in plethysms

- Construct explicit injective linear lifting map for $n \geq m$

$$\kappa^d_{m,n} \colon \mathrm{Sym}^d \mathrm{Sym}^m V \to \mathrm{Sym}^d \mathrm{Sym}^n V$$

- $\kappa^d_{m,n}$ defined as $d$-fold symmetric power of linear map

$$M \colon \mathrm{Sym}^m V \to \mathrm{Sym}^n V,\ p \mapsto p\, e_1^{n-m}$$

  multiplication with $e_1^{n-m}$, 1st standard basis vector $e_1 \in V = \mathbb{C}^N$

- Use duality to show for $f \in \mathrm{Sym}^d \mathrm{Sym}^m V$, $q \in \mathrm{Sym}^n V^*$,

$$\langle \kappa^d_{m,n}(f), q^d \rangle = \langle f, M^*(q)^d \rangle$$

  Here $M^* \colon \mathrm{Sym}^n V^* \to \mathrm{Sym}^m V^*$ denotes dual map of $M$.

- $M^*(q)$ is $(n-m)$-fold partial derivative of $q$ in direction $e_1$ (times $m!/n!$)

# Highest weight vectors in plethysms

- Proved that lifting

$$\kappa^d_{m,n} \colon \mathrm{Sym}^d \mathrm{Sym}^m V \to \mathrm{Sym}^d \mathrm{Sym}^n V \,,$$

  maps highest weight vectors of weight $\mu \vdash md$ to highest weight vectors of weight $\mu^{\sharp dn}$ ($\mu$ with extended 1st row)
- Constructed system of generators $v_T$ of space of highest weight vectors of weight $\mu$, labelled by tableaux $T$ of shape $\mu \vdash dm$ with $d$ letters, each occuring $m$ times (no letter appears more than once in a column)
- Proved: $\kappa^d_{m,n}$ maps generator $v_T$ to generator $v_{T'}$ where $T'$ arises from $T$ by adding in the first row $n - m$ copies of each of the $d$ letters
- Side result: new proof of known stability property of plethysms

# Corollary on lifting

**Cor. Lift**

Suppose $\lambda \vdash nd$ satisfies $\lambda_2 \leq m$ and $\lambda_2 + |\bar{\lambda}| \leq md$. Then every highest weight vector of weight $\lambda$ is obtained as a lifting.

Proof.

- $\lambda_2 + |\bar{\lambda}| \leq md$ is number of boxes of $\lambda$ that appear in non-singleton columns
- Hence $\lambda$ is obtained by extending the 1st row of some $\mu \vdash md$
- Let $T'$ be a tableau of shape $\lambda$ with $d$ letters, each occuring $m$ times. Since no letter appears more than once in a column, each of the $d$ letters appears at least $n - \lambda_2 \geq n - m$ times in singleton columns. Hence $T'$ is obtained from a tableau $T$ of shape $\mu$ as before
- From before: $\kappa^d_{m,n}(v_T) = v_{T'}$
- Moreover, the $v_{T'}$ generate space of hwv of weight $\lambda$ □

# Proof of Prop. ALL

**Prop. ALL**

$\lambda \vdash nd$ s.t. $|\bar{\lambda}| \leq md$ and $md^2 \leq n$. Then every highest weight vector of weight $\lambda$ in $\mathrm{Sym}^d\mathrm{Sym}^n V$ does not vanish on $\Omega_n$.

Proof.

- Let $h \in \mathrm{Sym}^d\mathrm{Sym}^n V$ be hwv of weight $\lambda$
- $\lambda_2 \leq |\bar{\lambda}| \leq md$ and $\lambda_2 + |\bar{\lambda}| \leq 2|\bar{\lambda}| \leq 2md \leq md \cdot d$
- Cor. Lift applied to $\mathrm{Sym}^d\mathrm{Sym}^{md} V \to \mathrm{Sym}^d\mathrm{Sym}^n V$ shows $h = \kappa^d_{md,n}(f)$ for some hwv $f \in \mathrm{Sym}^d\mathrm{Sym}^{md} V$ of weight $\lambda$
- Can show that for almost all power sums $p = \varphi_1^{md} + \cdots + \varphi_d^{md}$ we have $\langle f, p^d \rangle \neq 0$ and with $q := X_1^{n-md} p$,

$$\langle f, M^*(q)^d \rangle \neq 0$$

- Using duality

$$\langle h, q^d \rangle = \langle \kappa^d_{m,n}(f), q^d \rangle = \langle f, M^*(q)^d \rangle \neq 0.$$

By Prop. PPS, we have $q \in \Omega_n$ since $n \geq md \cdot d$. □

Thank you for your attention!