Geometric Complexity Theory
Lie Algebraic Methods for Projective Closures

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Outline

- Preliminaries, group actions - the stabilizer question.
- The Big Picture, structure, paths and our contributions
- The Local Model
  - Quotients of lie algebra action on $G \times_H V$.
  - The local model expressions and its properties.
- Forms - Limits and the local stabilizers
  - The family $f(t) = A(t) \cdot f = g + t^b f_b + \ldots$, with stabilizers $\mathcal{H}$ and $\mathcal{K}$ of g and f.
  - A basis for $\mathcal{K}(t)$ and its properties - $\mathcal{K}_0$ and $\overline{f_b}$.
  - The special case of $\lambda(t)$ and the triple-stabilizer conditions.
  - The co-dimension-1 case.
- One Matrix under Conjugation
  - The spectrum block partition and the top nilpotent. The local analysis of J_n.
- Advanced topics
  - The very local stabilizer $\mathcal{K}[\epsilon]$ and the Lie algebra cohomology condition.
  - Paths - Optimization and differential geometry.

Preliminaries

- G, connected reductive *algebraic* group over $\mathbb{C}$.
- $\rho : G \to End(V)$, rational representation.
- $y \in V$. *Orbit* of y , $O(y) := \{g \cdot y | g \in G\}$.
- $O(y)$ need not be closed, it is constructible.
- $\overline{O(y)}$, *orbit closure* of y - *Zariski* topology or *Euclidean topology*.
    - GL_n action on $\mathbb{C}^n$. $\overline{O(v)} = \mathbb{C}^n$, $v \in \mathbb{C}^n, v \neq 0$,
    - GL_n adjoint action on M_n. $\overline{O(J_n)} = \mathcal{N}$, the *nilpotent cone*.

Question:

- Given $x, y \in V$, is $x \in \overline{O(y)}$? Distinctive stabilizers.
- Given $[x], [y] \in \mathbb{P}(V)$, is $[x] \in \overline{O([y])}$? – direct relevance to algebraic complexity theory.

The Big Picture

We have $x, y \in V$, with a G-action, and λ such that:

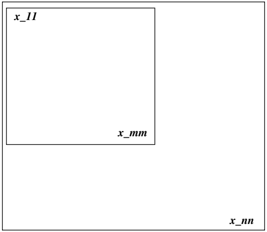
$$\lambda(t) \cdot y = t^a x + t^b y_b + \text{higher terms}$$

Thus x appears as the leading term in a weight-space decomposition of y under λ. How do we connect the stabilizers of x and y?

Example: $y = det(X_{nn})$ and $x = x_{nn}^{n-m} perm(X_{mm})$.

Question

Is there a homogenous substitution in the matrix so that the determinant becomes a smaller permanent, suitably homogenized?



Stabilizers change dramatically under taking limits!

The Big Picture

We have $x, y \in V$, with a G-action, and the question: Is there NO λ such that:

$$\lambda(t)y = t^a x + t^b y_b + \text{higher terms}$$

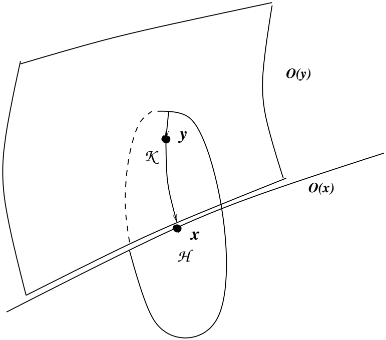
How do we determine the answer, even when x and y have distinctive stabilizers H, K?

The GCT approach: The distinctive stabilizers have the required information. The hunt for obstructions.

$$\mathbb{C}[\overline{O(y)}] \to \mathbb{C}[\overline{O(x)}] \to 0$$

- Representations as Obstructions. Burgisser, Ikenmeyer and others.
- Geometric Approach: Ressayre-Mignon-Landsberg and others. Algebro-geometric properties of y (or x) and their closures.
- Algebraic Complexity Theory: Various questions pertaining to orbits and closures, optimizability - Wigderson et. al.

The Stabilizer Problem -The Structure and the Path

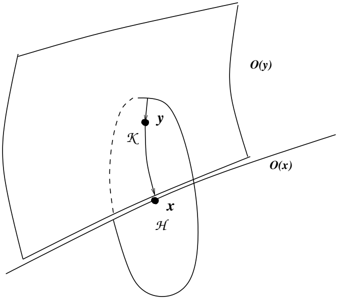


- Luna slice theorem: Rich structure of the neighborhood of x when it has a reductive stabilizer. No non-reductive version.
- Lie algebraic limits. If $\mathcal{K}$ is the stabilizer of q then $\lim_{t\to 0} \lambda(t)\mathcal{K}\lambda(t)^{-1} \subseteq \mathcal{H}$. But deeper connection with Lie algebra deformations needed.

What we achieve:

- build a Luna type partly factorized “lower-triangular” model for non-reductive points.
- illustrate the computability of this model and its connections with standard and classical problems.
- demonstrate that the local model retains (possibly) relevant differential geometry information in the vicinity of x.
- provides a beginning to understanding the deformation of the Lie algebra $\mathcal{K}$ (of y) to the Lie algebra $\mathcal{H}$ of x.
- provide a beginning to understanding the path along which $\mathcal{K} \rightsquigarrow \mathcal{H}$, the importance of the tangents of exit and entry.
- hopefully provide links, connections of GCT to geometry.

The Local Model

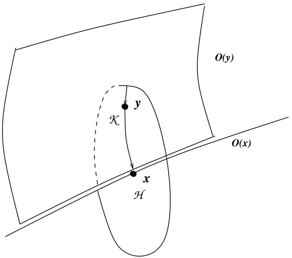


Stabilizer Question

- Luna: G action near $x \Leftrightarrow H$-action on N.
- Factorization of action - Stabilizers in the vicinity of x are subgroups of H, in fact stabilizers of $n \in N$.

The difficulty with extending Luna

- There is no N which is left invariant by H.
- A candidate is to use $\overline{N} = V/(T_x(O(x))$ and construct $G \times_H \overline{N} \to V$. But $\overline{N}$ is H-invariant. So that does not work.
- The only map which works is $G \times_H V \to V$ which has a non-zero-dimensional fiber. This is used to develop the model.

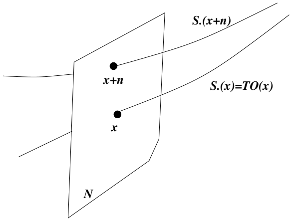


The Local Model

- Select a suitable $M \subseteq G$ and $N \subseteq V$ - the orbit and the normal.
- Let $\mathcal{S} \oplus \mathcal{H} = \mathcal{G}$. Let $M \subset G$, submanifold containing $e \in G$ and complementary to H at e such that $T_e M = \mathcal{S}$.
- $M \cdot x$ is the orbit, $M \cdot (x + n)$ is merely the same dimension of $M \cdot x$ and transversal to N. $\mathcal{S} \cdot (x + n) \oplus N = T_{x+n} V$.
- Let $O(x)$ be G-orbit of x. Let N be a complement to TO_x.

$$\mu : M \times N \to V : \mu(m, n) = m \cdot (x + n)$$

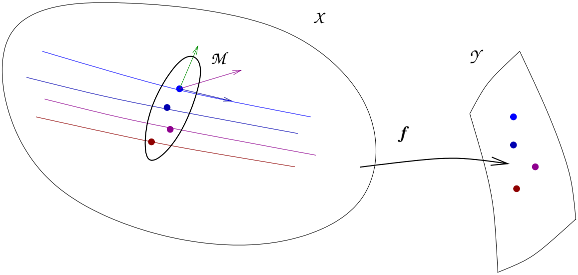
Clearly, μ is a local diffeomorphism.



The $\mathcal{G}$-action

Objective

- To obtain an explicit $\mathcal{G}$ action on a neighbourhood of $(e, 0) \in M \times N$ such that μ is equivariant.
- To compute this action on (e, N)



Immerse $M \times N$ in a suitable $\mathcal{G}$-equivariant map and project to $T(M \times N)$. Easy for Lie algebra actions!

The associated bundles

- The map $\mu : G \times V \to V$ given by $\mu(g, v) = g \cdot (x + v)$ factorizes through $G \times_H V$.

$$\mu(gh, h^{-1}v) = gh(x + h^{-1}V) = g(x + v) \text{ since } h \cdot x = x$$

$$\begin{array}{ccccc} G \times V & \to & G \times_H V & \to & V \\ & & \uparrow & \nearrow & \\ M \times N & \to & M \times V & & \end{array}$$

- Thus, $\mu : M \times V \to V$ is $\mathcal{G}$-equivariant and regular at $(e, 0)$.
- $M \times N$ is transversal to this map. This is the local model.

Recipe

- Compute the $\mathfrak{g}$-action on $M \times V$. Project that tangent vector to $M \times N$!

The Local Projections and the Action

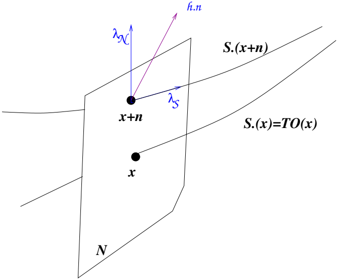
- For $\mathfrak{g} = \mathfrak{h} + \mathfrak{s}$ and $(e, n) \in M \times V$, we have:

$$\mathfrak{g} \cdot (e, n) = (\mathfrak{s}, \mathfrak{h} \cdot n)$$

- If $\mathfrak{h} \cdot n \in N$ then we are done! If not, let $\mathfrak{h} \cdot n = \mathfrak{s}' \cdot (x + n) + n'$, then:

$$\mathfrak{g} \cdot (e, n) = (\mathfrak{s} + \mathfrak{s}', n') = (\mathfrak{s} + \lambda_{\mathcal{S}}(n)(\mathfrak{h} \cdot n), \ \lambda_N(n)(\mathfrak{h} \cdot n))$$

- Thus $\mathfrak{g} \cdot (e, n) \in T_{(e,n)} M \times N = T_e M \times T_n N = \mathcal{S} \times N$.



The Local Model

The format - $\mathfrak{g} = \mathfrak{h} + \mathfrak{s}$ on the $\mathcal{S}$ part and the N part

element	$\mathcal{S}$	N
$\mathfrak{s}$	$\mathfrak{s}$	0
$\mathfrak{h}^{a} = \mathfrak{r} + \mathfrak{q}$	$\lambda_{\mathcal{S}}(\mathfrak{q} \cdot n)$	$\lambda_N(\mathfrak{h} \cdot n)$

[a] in a reductive-nilpotent Levi decomposition

- $\mathfrak{g} \cdot (e, n) = 0 \Rightarrow \lambda_N(\mathfrak{h} \cdot n) = 0$. The stabilizer condition: $\mathcal{H}_n = \{\mathfrak{h} | \lambda_N(\mathfrak{h} \cdot n) = 0\}$.
- For any $\mathfrak{h} \in \mathcal{G}$ and n, define its $\mathcal{S}$ completion as that element $\mathfrak{g} = \mathfrak{h} + \mathfrak{s}$ such that $\mathfrak{s} = -\lambda_{\mathcal{S}}(n)(\mathfrak{h} \cdot n)$. Then $\mathfrak{g} \cdot n = 0$.
- Note that $\mathcal{S}$-completion $\mathcal{H} \to \mathcal{G}$ is a linear operation.
- If $\mathcal{K} \subseteq \mathcal{G}$ is the stabilizer of $x + n$, then $\mathcal{H}_n \subseteq \mathcal{H}$ has same dimension and the $\mathcal{S}$-completion of every element of $\mathfrak{h} \in \mathcal{H}_n$ is $\mathcal{K}$.
- Note that reductive algebras may appear in the vicinity of nilpotent algebras.

We define the map: $\theta(n) : V \to V$ as:

$$\theta(n)(v) \triangleq \lambda_{\mathcal{S}}(0)(v) \cdot n$$

Thus, $T_x O(x)$ is both a subspace of V as well as a space $\mathcal{S}$ of lie algebra elements.
Note that $\theta(n) : V \to V$ is linear in n.

$$\lambda_{\mathcal{S}}(n) = \lambda_{\mathcal{S}} \circ (1 + \theta(n))^{-1} \text{ and } \lambda_N(n) = \lambda_N \circ (1 + \theta(n))^{-1}$$

Then, for $\mathfrak{g} = \mathfrak{h} + \mathfrak{s}$, we have:

$$\mathfrak{g} \circ (e, n) = (\mathfrak{s} + \lambda_{\mathcal{S}} \circ (1 + \theta(n))^{-1}(\mathfrak{h} \cdot n), \lambda_N \circ (1 + \theta(n))^{-1}(\mathfrak{h} \cdot n))$$

Equivariance

$H = RQ$ is the Levi decomposition and $\mathcal{R}$ and $\mathcal{Q}$ their lie algebras. Suppose that $\mathcal{S}$ and N are chosen as R-module complements of $\mathcal{H} \subseteq \mathcal{G}$ and $TO_x \subseteq V$. $\mathfrak{g} = \mathfrak{r} + \mathfrak{q} + \mathfrak{s}$, we have:

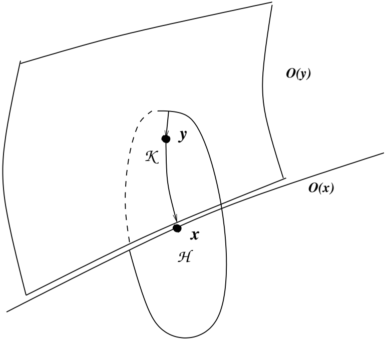
$$\mathfrak{g} \circ (e, n) = (\mathfrak{s} + \lambda_{\mathcal{S}} \circ (1 + \theta(n))^{-1}(\mathfrak{q} \cdot n), \mathfrak{r} \cdot n + \lambda_N \circ (1 + \theta(n))^{-1}(\mathfrak{q} \cdot n))$$

Moreover the $\Theta : V \otimes N \to V$ and $\Phi : \mathcal{S} \otimes N \to \mathcal{S}$ maps are R-equivariant.

$$\begin{aligned} \Theta(v \otimes n) &\triangleq \lambda_{\mathcal{S}}(v) \cdot n \\ \Phi(\mathfrak{s} \otimes v) &\triangleq \lambda_{\mathcal{S}}(\mathfrak{s} \cdot n) \end{aligned}$$

- Only the nilpotent part $\mathfrak{q}$ transmits to the $\mathcal{S}$-part. If $\mathcal{H}$ is reductive, we recover classical result.
- The "Gauss" map $\Phi : \mathcal{S} \times N \to \mathcal{S}$ is an algebraic version of curvature form $\Pi : \mathcal{S} \times \mathcal{S} \to N$. It captures the same data for reductive group representations.

Comparison



Luna	Local Model
G-action $\Leftrightarrow$ H-action on N	$\mathcal{H}$-action through λ_N and Θ
Horizontal action through G/H	Horizontal action through $\mathcal{Q}$ and $\mathcal{S}$.
Stabilizer $G_n \subseteq H$	$\mathcal{G}_n$ are $\mathcal{S}$-completions of $\mathcal{H}_n$
Valid for good orbits x	Valid for all x

Understanding $\mathcal{K} \rightsquigarrow \mathcal{H}$ in limits of forms.

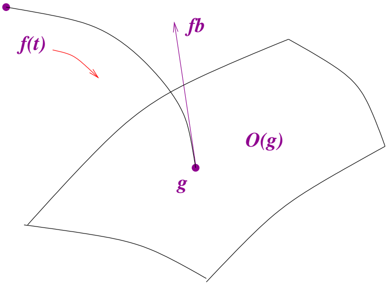
Forms

- Let X be a set of indeterminates and let V be the $GL(X)$-module $V = Sym^d(X)$ and $f, g \in V$ be non-zero forms.
- Let $\mathcal{G}$ be the algebra $gl(X)$, $\mathcal{K} \subseteq \mathcal{G}$ be the stabilizer of f and $\mathcal{H}$ be the Lie algebra stabilizing g.
- Algebraic family $A(t) \subseteq GL(X)$, parametrized by t such that $A(1) = e$, the identity element. Let

$$f(t) = A(t).f = t^a g + t^b f_b + t^{b+1} f_{b+1} + \ldots + t^D f_D,$$

- Such a family exists for projective closures in form space!

Forms

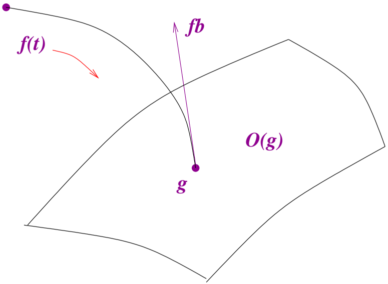


After re-normalization, we may assume:

$$f(t) = A(t).f = g + t^b f_b + t^{b+1} f_{b+1} + \ldots + t^D f_D,$$

with $f_b \neq 0$. We call f_b as the tangent of approach.
Transversality Assumption. Vector space spanned by $f_b, \ldots, f_D$ intersects $T_g O(g)$ trivially.

The Stabilizer Condition



- Notation: $A(t) \cdot f = f(t) = g + t^b f_b + \ldots$, and $f^+(t) = \sum_{i=b}^{D} f_i t^i$. Note that t^b divides $f^+(t)$.
- $\mathcal{K}(t_0)$ is the stabilizer of $f(t_0)$, then $\mathcal{K}(t_0) = A(t_0)\mathcal{K}A(t_0)^{-1}$. $dim(\mathcal{K}(t_0)) = dim(\mathcal{K}) = k$.
- Applying the local model to a parametrized family - $\mathcal{H}(t) \subseteq \mathcal{H}$ whose $\mathcal{S}$-completion is the stabilizer $\mathcal{K}(t)$.
- Gives us a uniform basis for $\mathcal{K}(t)$ which goes into $\mathcal{H}$ in the limit, giving us $\mathcal{K}_0$, a subalgebra of $\mathcal{H}$.

Key Proposition

1. There is a $\mathbb{C}(t)$-basis $\{\mathfrak{k}_i(t)\}_{i=1}^k$ of $\mathcal{K}(t)$, the stabilizer Lie algebra of $f(t)$ and a large number D such that

$$\mathfrak{k}_i(t) = \sum_{j=0}^{D} (\mathfrak{s}_{ij} + \mathfrak{h}_{ij}) t^j, = \mathfrak{h}_i(t) + t^b \mathfrak{s}_i(t)$$

   for suitable elements $\mathfrak{s}_{ij} \in \mathcal{S}, \mathfrak{h}_{ij} \in \mathcal{H}$. The dimension of $\mathcal{K}_0$ is the same as the dimension of $\mathcal{K}(t_0)$ for a generic $t_0 \in \mathbb{C}$.
2. Let $\mathcal{H}(t) = \mathbb{C}(t) \cdot \{\mathfrak{h}_i(t) | i = 1, \ldots, k\}$. For any generic $t_0 \in \mathbb{C}$, subspace $\mathcal{H}(t_0) \subseteq \mathcal{H}$, has dimension k and $\mathcal{K}(t_0)$ is the $\mathcal{S}$-completion of $\mathcal{H}(t_0)$ for the point $f^+(t_0)$.
3. The space $\mathcal{K}_0 = \mathbb{C} \cdot \{\mathfrak{k}_1(0), \ldots, \mathfrak{k}_k(0)\}$, $\mathbb{C}$-space of leading terms $\mathfrak{k}_i(0) = \mathfrak{k}_{i0} = \mathfrak{h}_{i0}$ is a Lie subalgebra of $\mathcal{H}$ and of dimension k. Moreover, if $\mathfrak{k}(t) \in \mathcal{K}(t)$ is any element such that $\mathfrak{k}(0)$ is defined, then $\mathfrak{k}(0) \in \mathcal{K}_0$.

The Wrap-up Theorem

The $\star$-action: $\mathcal{H}$ acts on $\overline{N} = V/(TO(g))$. This is the traditional quotient action. If H is reductive, N may itself be an $\mathcal{H}$-module.

1. For any element $\mathfrak{h} \in \mathcal{K}_0$, we have $\lambda_N(\mathfrak{h} \cdot f_b) = 0$ and thus, there is an $\mathfrak{s} \in \mathcal{S}$ such that $\mathfrak{s} \cdot g + \lambda_{\mathcal{S}}(\mathfrak{h} \cdot f_b) = 0$. Paves the way for $\star$-action
2. The subalgebra $\mathcal{K}_0$ is contained within $\mathcal{H}_b \subseteq \mathcal{H}$, the stabilizer of $\overline{f_b}$ for the $\star$-action of $\mathcal{H}$.
3. The local model matches $G \times_H \overline{N}$ infinitesimally close to x.

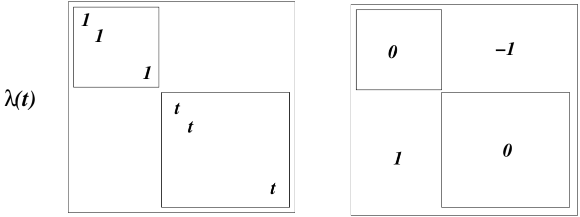
The coupling of $\mathcal{K}$ and $\mathcal{H}$

$$\mathcal{K} \rightsquigarrow \mathcal{K}_0 \to \mathcal{H}_b \to \mathcal{H}$$

Thus, when the small permanent appears in the orbit closure of the determinant, a "smashed" version of the stabilizer of the determinant sits inside the stabilizer of the homogenized small permanent as a subalgebra!

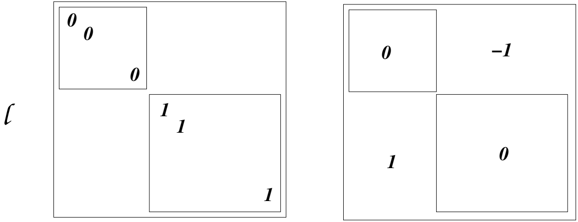
Important, in GCT as well as in Kempf and other limits.

$$\lambda(t) \cdot f = t^a g + t^b f_b + \text{ higher terms}$$



- Implements singular substitutions $f(A \cdot X)$. Let $X = Y \cup Z$. $\lambda(t) \cdot y = y, \forall y \in Y$, $\lambda(t) \cdot z = tz, \forall z \in z$
- $\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$. $\lambda(t)\mathfrak{g}_i\lambda(t)^{-1} = t^i \mathfrak{g}_i$ for any $\mathfrak{g}_i \in \mathcal{G}_i$, and that $\mathcal{G}_i \cdot V_j = V_{i+j}$.

The 1-PS case



- $\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$. $\lambda(t)\mathfrak{g}_i\lambda(t)^{-1} = t^i\mathfrak{g}_i$ for any $\mathfrak{g}_i \in \mathcal{G}_i$, and that $\mathcal{G}_i \cdot V_j = V_{i+j}$.
- $\mathcal{H}$ is graded. We can choose $\mathcal{S}_i \subseteq \mathcal{G}_i$ to be a complement of $\mathcal{H}_i$, so that $\mathcal{S}$ is graded. $\mathcal{H}_b$ is graded.
- $\mathcal{K}_0$ are the leading terms of elements of $\mathcal{K}$. So $\mathcal{K}_0$ is graded.

The Lie algebra element ℓ

- Setting $\ell = \log(t^{-a/d}\lambda(t))$, (i) $\ell \in \mathcal{H}$, $\ell \notin \mathcal{H}_b$ but $[\ell, \mathcal{H}_b] \subseteq \mathcal{H}_b$ and $[\ell, \mathcal{K}_0] \subseteq \mathcal{K}_0$, ℓ normalizes $\mathcal{H}_b, \mathcal{K}_0$.

The tangent of exit

Definition

Assume $\lambda(t)f = t^a g + t^b f_b + \cdots + t^D f_D$. Then the tangent of exit is the form $\lim_{t\to 1} \frac{f(t)-f(1)}{t-1}$.

- $\ell f = af_a + bf_b + \ldots + Df_D$, and the tangent of exit is $\ell f - f$.
- ℓf is in an element of TO_f.
- Tangent of exit given by the action of ℓ, $\ell \notin \mathcal{K}$.

Key Questions

- What is the stabilizer of ℓf within TO_f, under $\mathcal{K}$?
- Are there some symmetries which $f, \ell f, g$ share, and will that shed light on $\mathcal{K} \rightsquigarrow \mathcal{H}$?
- **Element $\mathfrak{k} \in \mathcal{K}$ of pure ℓ-weight are triple stabilizers!**

Let f, g, λ and ℓ be as above. Then at least one of the following hold:

(A) $\mathcal{K}_0$ is a nilpotent algebra, or

(B) there is a unipotent element $u \in U(\lambda)$ and an element $\mathfrak{k} \in \mathcal{K}$ such that g is a limit of f^u under λ and $\mathfrak{k}^u$ is a triple stabilizer for the data $(f^u, \ell f^u, g)$.

semi-simple elements in $\mathcal{K}$ of ℓ-weight zero! Or more generally:

$$\mathfrak{k}^u \in \mathcal{K}^u(\rightsquigarrow \mathcal{K}_0^u) \to \mathcal{H}_b^u \to \mathcal{H}$$

Closure of affine forms

Let g be a projective limit of codimension 1 of f under λ as above. Then at least one of the following conditions hold:

1. $\mathcal{K}_0$ is nilpotent and $\mathcal{H} = \mathcal{K}_0 \oplus \mathbb{C}\ell$ is a Levi factorization of $\mathcal{H}$.
2. There are unipotent elements $u \in U(\lambda)$ and elements $\mathfrak{k} \in \mathcal{K}$ such that $u\mathfrak{k}u^{-1}$ stabilizes g, f^u and ℓf^u, and g is the limit of f^u under λ.

$$det_3(X) = det\left(\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}\right)$$

$1-PS$	forms	$dim((\mathcal{K}_0)_i)$ $dim((\mathcal{H})_i)$			ℓf	$dim(\mathcal{K}_{\ell f})$
	(f,g)	1	0	-1		
ℓ_1	det_3	0	8	8		$-$
	Q_1	0	$8+1$	8	Q_1'	$0+4+0$
ℓ_2	det_3	0	8	8		$-$
	Q_2	0	$8+1$	8	Q_3	$0+8+0$
ℓ_4	det_3	1	10	5		$-$
	Q_4	1	$13+1$	7	Q_4'	$1+6+1$

Matrices under Conjugation

Conjugation action..

Jordan Form

Affine closures well understood, both inside and outside the null-cone. How do points y outside the null-cone interact with x inside?

Projective orbit closures

- Let x be a nilpotent matrix completely characterized by its nilpotent-block partition.
- Let y be any matrix described by its Jordan canonical form.
- When does x belong to the projective orbit closure of y?

When y itself is nilpotent, this is determined by the dominance ordering on the nilpotent-block partitions.

The Krull-Schmidt Dimensions

The dimensions of Krull-Schmidt sequence determines projective closures.

Spectrum block-partition

Let y be in Jordan canonical form with s eigenvalues $\mu_1, \ldots, \mu_s$.
For μ_i, we define the block-partition $\lambda_i = (\lambda_{i1} \geq \lambda_{i2} \geq \ldots)$ which records the sizes of Jordan blocks of type μ_i
The spectrum block-partition λ of y is the sum $\lambda = \lambda_1 + \ldots + \lambda_s$

$$y = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

The block-partitions are $(2, 1, 1)$, (2) for eigenvalues -1 and 3
The spectrum block-partition of y is $(2, 1, 1) + (2) = (4, 1, 1)$

If y has m distinct eigenvalues, its spectrum block-partition is (m).

Projective orbit closure theorem

Main Theorem

Let x be a nilpotent matrix whose nilpotent block-partition is θ. and y be a matrix whose spectrum block-partition is λ
The projective orbit closure of y contains x iff $\theta \trianglelefteq \lambda$.

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad y' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- The spectrum block-partition of y is $(1,1) + (1) = (2,1)$
- The spectrum block-partition of y' is $(2) + (1) = (3)$.
- The nilpotent block-partition of x is (3)

We conclude that

- $y \in \overline{O(y')}$ (classical)
- $[x] \notin \overline{O([y])}$ (our theorem)
- $[x] \in \overline{O([y'])}$ (our theorem)

Computability of local model at J_m

The induced Lie algebra $\mathcal{G}$-action $\mathfrak{g}.A = [\mathfrak{g}, A] = \mathfrak{g}A - A\mathfrak{g}$
We fix J_m and apply the local model at J_m

$$J_m = \begin{bmatrix} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & 0 & \dots \\ & \vdots & & \vdots & \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

The relevant data for J_4 is as follows

$$\mathcal{H} = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha & \beta & \gamma \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \alpha \end{bmatrix} \mathcal{S} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} N = \begin{bmatrix} * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix}$$

Stabilizers in the neighborhood of J_m

We fix $x + n = J_m + n$ where $n \in N$. Say

$$x + n = \begin{bmatrix} -c_3 & 1 & 0 & 0 \\ -c_2 & 0 & 1 & 0 \\ -c_1 & 0 & 0 & 1 \\ -c_0 & 0 & 0 & 0 \end{bmatrix}$$

This is the familiar **companion form**. Next, for $\mathfrak{g} = \mathfrak{h} + \mathfrak{s}$

$$\mathfrak{g}.(x + n) = (\mathfrak{s} + \lambda_{\mathcal{S}} \circ (1 + \theta(n))^{-1}(\mathfrak{h} \cdot n), \lambda_N \circ (1 + \theta(n))^{-1}(\mathfrak{h} \cdot n))$$

Calculations:

$$\theta^i(n) = 0 \;\; \forall i \geq 2 \text{ and } \lambda_N \circ (1 - \theta(n))(\mathfrak{h} \cdot n)) = 0$$

So, the stabilizer condition is: $\mathfrak{s} = -\lambda_{\mathcal{S}} \circ (1 - \theta(n))(\mathfrak{h} \cdot n)$

In other words, the stabilizer is the $\mathcal{S}$-completion of the whole subspace $\mathcal{H}$ and has dimension m.

Diagonalizable matrices with J_m in the projective closure

Let y be a polystable point of the form $x + n$ whose projective closure contains $x = J_m$

$$y = x + n = \begin{bmatrix} -c_3 & 1 & 0 & 0 \\ -c_2 & 0 & 1 & 0 \\ -c_1 & 0 & 0 & 1 \\ -c_0 & 0 & 0 & 0 \end{bmatrix}$$

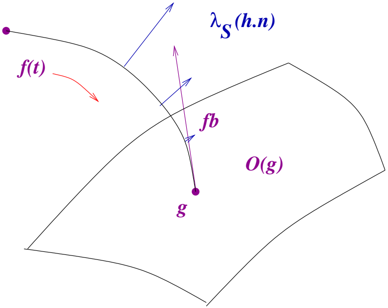
As y is diagonalizable and has stabilizer dimension m, this forces that y has distinct eigenvalues!

Observation: The minimal polynomial of $x + n$ is also its characteristic polynomial, namely $z^m + c_{m-1}z^{m-1} + \ldots + c_0$

Advanced topics

Motivating the infinitesimal analysis

Understanding $\lim_{t\to 0}\mathcal{K}(t)=\mathcal{K}_0$.



The asymmetry in the degree of f

For any element $\mathfrak{g}=t_0^b\mathfrak{s}+\mathfrak{h}\in\mathcal{K}(t_0)$ we have:

$$\mathfrak{s}+\lambda_{\mathcal{S}}((1+\theta(f^+(t_0)))^{-1}((\mathfrak{h}\cdot f^+(t_0)))=0. \tag{1}$$

Formulating the local stabilizer

- $\mathfrak{h} + \epsilon\mathfrak{s}$ stabilizes $p = g + \epsilon f_b$, with $\epsilon^2 = 0$!
- $\mathcal{G}[\epsilon] = \mathcal{G} \otimes \mathbb{C}[\epsilon]$, similarly $\mathcal{H}[\epsilon]$
- Let $\mathcal{H}_p \subseteq \mathcal{G}[\epsilon]$ is the stabilizer of p. **Model for $\mathcal{K}(t)$?**
- Given any $\mathfrak{h} \in \mathcal{H}_b$, $\mathfrak{h} \cdot f_b$ is a tangent vector TO_g. Define: $d_b : \mathcal{H}_b \to \mathcal{G}/\mathcal{H}$. **$d_b$ is a derivation.**
- The stabilizer $\mathcal{H}_p \subseteq \mathcal{G}[\epsilon]$ is given by the set $\mathfrak{h}_0 + \epsilon\mathfrak{g}$ such that (i) $\mathfrak{h}_0 \in \mathcal{H}_b$, and (ii) $\mathfrak{g} \in d_b^{-1}(-\mathfrak{h}_0 f_b)$.
- $\epsilon\mathcal{H} \subseteq \mathcal{H}_p$. **$\mathcal{H}_p$ is too big.** $\mathbb{C}[\epsilon]$-dimension $> k$.

The extension question

Is there an $\overline{\mathcal{K}}$ of $\mathbb{C}[\epsilon]$- dimension k which stabilizes p and such that

$$\overline{\mathcal{K}} \xrightarrow{\epsilon=0} \mathcal{K}_0$$

The extension problem

The existence of $\overline{K}$ is equivalent to the existence of a derivation $\overline{d_b}$ extending d_b.

$$
\begin{array}{ccc}
\mathcal{G}/\mathcal{K}_0 & \longrightarrow & \mathcal{G}/\mathcal{H} \\
\uparrow \overline{d_b} & & \uparrow d_b \\
\mathcal{K}_0 & \longrightarrow & \mathcal{H}_b
\end{array}
$$

Theorem: Let g be a regular limit of f via $A(t)$ and $f(t)$ and f_b be as above. Let $\mathcal{K}(t)$ be the stabilizing Lie algebra of $f(t)$ and $\mathcal{H}$ that of g. Moreover, let $\mathcal{K}_0 \subseteq \mathcal{H}_b$ be the limit of $\mathcal{K}(t)$, as $t \to 0$. Let $d_b : \mathcal{K}_0 \to \mathcal{G}/\mathcal{H}$ be the derivation as above. Then there is a derivation $\overline{d_b} : \mathcal{K}_0 \to \mathcal{G}/\mathcal{K}_0$ which extends d_b.

We have the exact sequence of $\mathcal{K}_0$-modules:

$$0 \longrightarrow \mathcal{H}/\mathcal{K}_0 \longrightarrow \mathcal{G}/\mathcal{K}_0 \longrightarrow \mathcal{G}/\mathcal{H} \longrightarrow 0$$

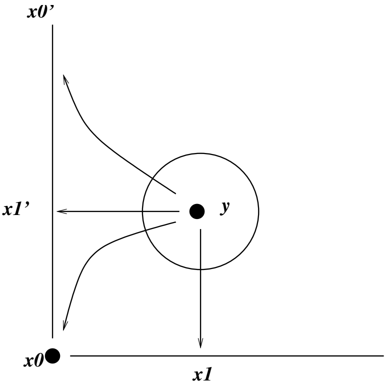
and the corresponding long exact sequence of cohomology modules:

$$\begin{array}{rcl} 0 & \longrightarrow & H^0(\mathcal{K}_0, \mathcal{H}/\mathcal{K}_0) \longrightarrow H^0(\mathcal{K}_0, \mathcal{G}/\mathcal{K}_0) \longrightarrow H^0(\mathcal{K}_0, \mathcal{G}/\mathcal{H}) \\ & \longrightarrow & H^1(\mathcal{K}_0, \mathcal{H}/\mathcal{K}_0) \longrightarrow H^1(\mathcal{K}_0, \mathcal{G}/\mathcal{K}_0) \longrightarrow H^1(\mathcal{K}_0, \mathcal{G}/\mathcal{H}) \\ & \longrightarrow & H^2(\mathcal{K}_0, \mathcal{H}/\mathcal{K}_0) \longrightarrow \dots \end{array}$$

Since both d_b and $\overline{d_b}$ are derivations, they belong to the spaces $H^1(\mathcal{K}_0, \mathcal{G}/\mathcal{H})$ and $H^1(\mathcal{K}_0, \mathcal{G}/\mathcal{K}_0)$ respectively (but they may be 0).

- Variation of the ideas of Nijenhuis and Richardson where the 2-cocyles are the infinitesimal directions of deformations.
- Likely that $\mathcal{K}_0$ is not *rigid* while $\mathcal{K}(t_0)$ are, for generic $t_0 \in \mathbb{C}$.

- Already a close connection between Π, i.e., local curvature form and Θ, the local model.
- This generalizes to higher forms as well. The curvature at y is easily expressed in terms of ρ.
- Different starting directions lead to different limit points. Kempf gives us one optimization function.
- Are their other local functions at y?



Conclusion, open questions

1. What is the "null cone" in the local model. For Luna, it is the null cone for the H action on N.
2. Is there an algebraic geometry-analogue of the local model?
3. Can we classify "generic" limits and their nilpotent algebras $\mathcal{K}_0$?
4. What limits do nilpotent families e^{At} give rise to? How do limit points accumulate stabilizing elements?
5. Can "generic" co-dimension-1 boundary forms exist for the determinant?
6. Are the cohomological conditions effective and salient? Especially, in the co-dimension 1 case?
7. The non-uniqueness of the optimization problem seem to be the core of the projective closure problem. Is there a unification of the Kempf's 1-PS theory and "optimal" paths?

Thank You!