Actions on Varieties, Representations, Chevalley's Theorem

Q. What is an algebraic group?

A. It's a group object in the category of varieties.

But what is a group object? Suppose we have a category---e.g., sets and maps (SET); topological spaces and continuous maps (TOP); varieties and morphisms (VAR)

Assume: finite products and final object C^0 exist in this category.

Product here means: usual Cartesion product of sets; Y x Z with the product topology, if Y, Z are top. spaces

But this is too loose and not good enough for our present purposes. We need to understand this better.

$$
\begin{tikzcd}
A \arrow[dr, dotted, "g"] \arrow[drr, bend left, "g_Z"] \arrow[ddr, bend right, "g_Y"'] & & \\
& Y \times Z \arrow[r] \arrow[d] & Z \\
& Y &
\end{tikzcd}
$$

Definition of **Product by its Universal Property**

Given objects Y and Z, by **YxZ** we mean an object (along with maps YxZ $\xrightarrow{p_Y}$ Y and YxZ $\xrightarrow{p_Z}$ Z in the category, although usually these are tacitly understood and not always written)

such that these maps are "universal" in the following sense: given any object A and maps A $\xrightarrow{g_Y}$ Y and A $\xrightarrow{g_Z}$ Z there exists a unique map A $\xrightarrow{g}$ YxZ such that $\pi_Y \circ g = g_Y$, $\pi_Z \circ g = g_Z$

In other words, to give a morphism to YxZ from some object A is equivalent to giving two morphisms: one from A to Y and another from A to Z.

Note: YxZ = ZxY

A group object: an object G equipped with three maps:

$G \times G \xrightarrow{\mu} G$ "multiplication"; $G \xrightarrow{i} G$ "inverse"; $\{*\} \xrightarrow{1} G$ "identity"

such that the usual axioms hold

$$\begin{tikzcd}
(G \times G) \times G \arrow[r, "\mu \times id"] \arrow[d, "id \times \mu"'] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"'] \arrow[ur, phantom, "\circlearrowleft"] & G
\end{tikzcd}$$

associativity

$G \simeq \{*\} \times G \simeq G \times \{*\}$

$$\begin{tikzcd}
G \arrow[r, "1 \times id"] \arrow[d, "id \times 1"'] \arrow[dr, "id"] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"'] & G
\end{tikzcd}$$

$g = 1g = g1$

$$\begin{tikzcd}
G \arrow[rr, "id \times i"] \arrow[d, "i \times id"'] \arrow[r] & \{*\} \arrow[dr, "1"] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[rr, "\mu"'] & & G
\end{tikzcd}$$

inverse

$g^{-1}g = gg^{-1} = 1$

Note: An object defined by a universal property is unique up to unique isomorphism.

$$\begin{tikzcd}
X \times Y \times Z \arrow[d, "p_X"'] \arrow[dr, "p_Y"] \arrow[drr, "p_Z"] & & \\
X & Y & Z
\end{tikzcd}$$

if it exists!

Upshot: $(X \times Y) \times Z \simeq X \times (Y \times Z) \simeq X \times Y \times Z$

Example: Group objects in SET are precisely groups.

Group objects in TOP are called Topological Groups.

Any topological group is also just a group (forgetting the topology).

$GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $\underbrace{M_n(\mathbb{C})}_{\text{addition}}$ are all topological groups.

Category of varieties and morphisms (VAR): The precise definition will take a while! But we can say a few things for now.

Whatever a variety is, it has the following attributes:

an underlying set of "closed points".

a topology on this set the Zariski topology.

a ring of functions called the regular functions.

Let $X \subseteq \mathbb{C}^n$ be an affine algebraic variety. That is, X is zero locus of a set of polynomials (f_i | i in I) in $\mathbb{C}[x_1, \dots, x_n]$

some index set

E.g. nxn matrices of determinanat 1

$\mathbb{C}[x_1, \dots, x_n] / \sqrt{(f_i \mid i \in I)}$

the parabola defined by $y=x^2$

Underlying set is the loci of zeros: $SL(n, \mathbb{C})$ in the first case

Topology is the subspace topology of the Zariski topology on the ambient affine space (closed sets are zero loci of sets of polynomials).

Ring of functions : $\mathbb{C}[x_{11}, \dots, x_{nn}] / (\det - 1)$; $\mathbb{C}[x, y] / (y - x^2)$

Product of two affine algebraic varieties: $Y \subseteq \mathbb{C}^m$; $Z \subseteq \mathbb{C}^n$

Y defined by vanishing of f_i: $f_i \in \mathbb{C}[y_1, \dots, y_m]$ $(i \in I)$

Z defined by vanishing of g_j: $g_j \in \mathbb{C}[z_1, \dots, z_n]$ $(j \in J)$

YxZ $\subseteq \mathbb{C}^m \times \mathbb{C}^n$ defined by f_i, g_j: $k[Y \times Z] = \mathbb{C}[\underline{y}, \underline{z}] / \sqrt{(f_i, g_j)}_{i \in I, j \in J}$

Caveat: Underlying set of the variety YxZ is YxZ

The topology of the variety YxZ is NOT the product topology

Example: $\mathbb{C} \times \mathbb{C} \simeq \mathbb{C}^2$ with its Zariski topology

But the Zariski topology on $\mathbb{C}^2$ is not the product of the Zariski topologies on $\mathbb{C}$.

Morphisms between affine varieties $Y \subseteq \mathbb{C}^m, \ Z \subseteq \mathbb{C}^n$

$$\mathbb{C}^m \xrightarrow{f} \mathbb{C}^n$$
$$(y_1, \dots, y_m) \quad (z_1, \dots, z_n)$$

$z_j(f(\underline{y}))$ is a poly in $\underline{y}$ $\forall\, 1 \le j \le n$

Morphisms from Y are just restrictions of morphisms from $\mathbb{C}^m$

Morphisms to Z are those which have image in Z.

$$Y \to Z$$
$$k[Y] \leftarrow k[Z]$$

Attributes of morphisms:

There is an underlying map of sets.

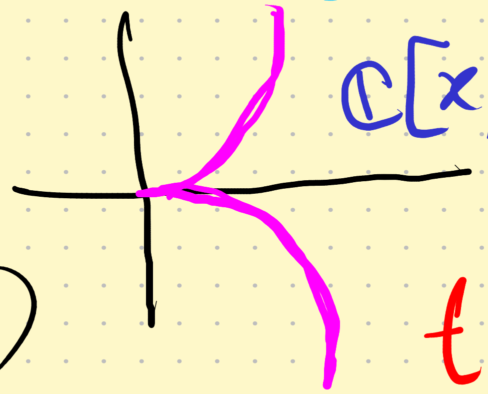
The above map is continuous in the Zariski topology.

Regular functions pull back to regular functions.

Example: (of a morphism that is bijective but not invertible)

$Y \subseteq \mathbb{C}^m$ $\mathcal{I}(Y)$ is a radical ideal $k[Y] = \dfrac{\mathbb{C}[y_1, \dots, y_m]}{\mathcal{I}(Y)}$

$$\mathbb{C} \longmapsto \mathbb{C}^2$$
$$t \longmapsto (\underset{x}{t^2}, \underset{y}{t^3})$$



$\mathbb{C}[x,y]$

$$t = \frac{t^3}{t^2} = \frac{y}{x}$$

Action of an algebraic group G on a variety X: a morphism $G \times X \dashrightarrow X$, which at the level of points defines an aciton

that is, $g(hx)=(gh)x$ and $1x=x$

Example: G on itself in multiple ways: left, right, conjugation, twisted conjugation, ...

Affine Algebraic Group (= Linear Algebraic Groups, as we will see)

An algebraic group whose underlying variety is affine.

Alternatively, a group object in the category of affine varieties.

Example: $\mathbb{C}^{n}, \mathbb{C}^{n,+}$, $SL(V)=SL_n(\mathbb{C}) \subseteq \mathbb{C}^{n^2}$ $(n=\dim V)$

$k[\mathbb{C}^n]=$ polynomial ring in n variables

$k[SL_n(V)]=$ poly. ring in n^2 variables / (det -1)

Recall: If f is a non-constant polynomial (in n variables), then D(f)= points (in $\mathbb{C}^n$) where f does not vanish (which is a Z-open set) is considered as an affine variety by the following identification:

$(\underline{x}, y) \subseteq \mathbb{C}^{n+1}$ satisfying $f(x)y=1$ <--> D(f) (forgetting y)

$$k[D(f)] = \mathbb{C}[\underline{x}, y]/(f(\underline{x})y-1) \simeq \mathbb{C}[\underline{x}, \tfrac{1}{f}]$$

More examples: $(\mathbb{C}^\times)^n$, GL(V), abstract subgroups of GL(V) that are defined by vanishing of (regular) functions: diagonal, upper Δ^r, ...

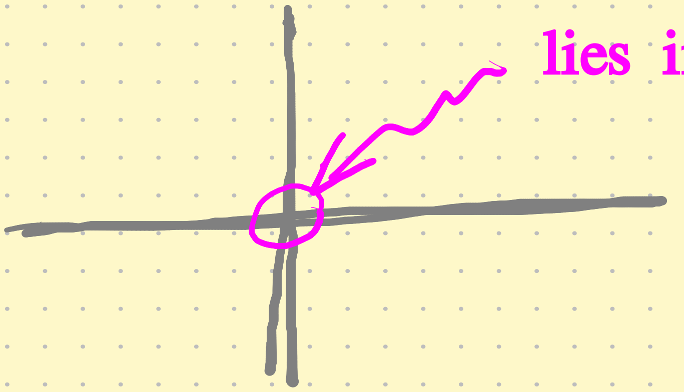
Representation

"Defining Actions" $GL(V) \times V \dashrightarrow V$

For g in G, $v \dashrightarrow gv$ is a linear map of V

Representation of an algebraic group G: $G \dashrightarrow GL(V)$ morphism such that at the level of points it is a group homorphism

Irreducibility: Points in a variety typically display different algebro-geometric behaviour. E.g. $xy=0$



lies in two irreducible components
as opposed to other points

Recall: A subset X of a topological space is irreducible if:
C, D closed and $X \subseteq C \cup D \Longrightarrow X \subseteq C$ or $X \subseteq D$.

Facts: Irreducible implies connected; Singletons are irreducible; Closure of an irreducible set is irreducible; Continuous image of irreducible sets are irreducible; Any two non-emtpy open sets of an irreducible space meet non-trivially; Non-empty open sets of an irreducible space are dense.

FACT: Let X be a variety. There are fintiely many maximal irreducible subsets of X. These are closed and cover X.

These maximal irreducible (closed) subsets are the irreducible components

Homogeneity: Every point of an algebraic group (more generally every point of a variety on which such a group acts morphically and transitively) displays the same algebro-geometric (hence topological) behaviour.

$G \times X \to X$

$\cup\!\!\mid$

$\{g\} \times X$

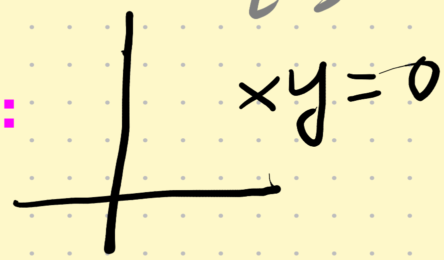
$\wr\!\wr$

X

Because g: X --->X given by x--->gx is an isomorphism

g-inverse would be its inverse.

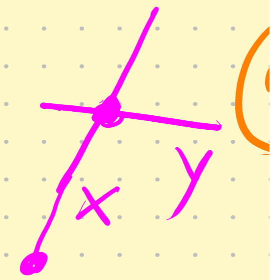
$\{x\} \subseteq$ ——

Example of the usage of this: 

is connected but not irreducible

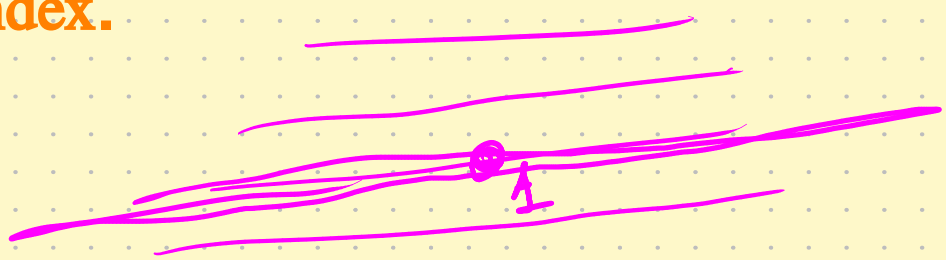
A connected algebraic group is irreducible

Proof: Suppose there is more than one irreducible component. If no two of them intersect, we would get a disconnection. If two of them--say X and Y---intersect, then a point in the the intersection would be contained in more than one irreducible component. But there is also a point contained in precisely one component, eg., any in X \ (union of the other components). $\not\Rightarrow\not\Leftarrow$



Proposition: The (finitely many) irreducible components of an algebaic group G are pairwise disjoint. The identity component is a normal subgroup of finite index.

Proof: Pleasant exercise.



$X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n$

Dimension of a variety: Maximum length of a chain of proper inclusions of irreducible closed subsets. (one of many eqv. defns.)

Corollary: A proper closed set of an irreducible variety has smaller dim.

Theorem: Supose that an algebraic group G acts on a variety X. Let Y be the closure of an orbit Gx (x in X). Then:

(i) Y is irreducible if G is connected ✓

(ii) Gx is open in Y

(iii) Y contains a closed orbit of G.

(iv) Y is also the usual closure of Gx.

$G \to X$, $g \mapsto gx$

$Gx \subseteq \overline{Gx} = Y$

Proof: Let us assume G is connected (eq. irreducible). The general case is easily deduced from the special case using the Proposition.

(i) Gx is irreducible (continuous image), Y is the closure of Gx

(ii) Uses homogeneity and the following

(Chevalley) **FACT:** The image of a morphism contains a non-empty open subset of its closure.

Move the open set around in the orbit Gx to see it is open in Y.

(iii) Y is a union of orbits. Y\Gx has smaller dimension. Choose an orbit in Y\Gx and keep going. Eventually (since the dimension keeps going down) we will get an orbit that is closed.

(iv) Follows from FACT: Suppose we have a ~~non-empty~~ not needed Zariski open set U of a Zariski closed set in $\mathbb{C}^n$. Then the usual clossure of U equals the Zariski closure of U.

QED

Example: GL_n acts on M_n by left multiplication

Orbits are determined by: row reduced echelon forms

The zero matrix forms a closed orbit.

Example: GL_n acts on M_n by conjugation

Orbits are indexed by: Jordan Canonical Forms

Scalar matrices form singleton orbits, hence closed.

The orbit of any matrix M with distinct eigenvalues is closed

Because N belongs to this orbit if and only if it has the same characteristic polynomial as M.

Example: $GL_n \times GL_n$ acts on M_n

$$(g,h)\cdot X = gXh^{-1}$$

why? To make it a left action

Orbits are indexed by: rank

The zero matrix is an orbit by itself, hence closed

Notation $G \circlearrowleft X$ means G acts on X

Suppose $G \circlearrowleft X$. Then G acts on all objects associated with X

Suppose $G \circlearrowleft X$ and $G \circlearrowleft Y$. Then G acts on anything constructed naturally from X and Y.

$$G \circlearrowleft X \implies G \circlearrowleft k[X]: \quad ({}^g f)(x) = f(g^{-1}x)$$

Why is ${}^g f$ in k[X]? (for g in G and f in k[X])

$$\begin{aligned} X &\to X \\ x &\mapsto g^{-1}x \end{aligned}$$
is a morphism and ${}^g f$ is the pull back of f under this.

g acts on k[X] as an algebra automorphism: ${}^g(f+h) = {}^g f + {}^g h$,

${}^g\lambda = \lambda$ (for scalar functions λ), ${}^g(fh) = {}^g f\, {}^g h$.

In particular it acts as a linear automorphism: g^{-1} is the inverse.

Theorem: Each f in k[X] is contained in a finite dimensional G-invariant subspace V of k[X]. For such a V, the resulting group homomorphism G-->GL(V) is a morphism.

Proof: GxX to X morphism. Pull back of functions:

$$k[G] \otimes k[X] \longleftarrow k[X] \qquad \underbrace{\sum \alpha_i \otimes \beta_i}_{\text{finite sum}} \longleftarrow\!\shortmid f$$

Let's compute ${}^g f$.

$${}^g f(x) = f(g^{-1}x) = f^*(g^{-1}, x) = \sum_i \alpha_i(g^{-1})\beta_i(x)$$

$${}^g f = \sum_i \alpha_i(g^{-1})\beta_i$$ So f is in the linear span of β_i

${}^g f \in \langle \beta_i \rangle \ \forall g \in G.$

Thus $\underbrace{\langle {}^g f \mid g \in G \rangle}_{\text{finite dimensional}} \subseteq \langle \beta_i \rangle$

Second Assertion: We will write GL(V) as GL(n) (where n = dim V) for an appropriate choice of basis of V. Each element x of X defines a linear functional δ_x on V: $\delta_x(f) := f(x)$.

As x varies over X, these functionals span V*. Suppose not. Then there exists non-zero f in V such that $\delta_x(f)$= f(x)=0 for all x.

Choose $x_1, x_2, \ldots, x_n$ in X such that they form a basis for V*. Let $f_1, \ldots, f_n$ be the dual basis.

Identify GL(V) with GL(n) using this basis. A simple calculation shows that the ij^th entry of the image of g under G-->GL(n) is $f_j(g^{-1}x_i)$ which is regular on G.