REPRESENTATIONS IN COORDINATE RINGS, GL_n REPRESENTATIONS

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1. Summary

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Algebraic geometry could be considered as the science of solving polynomials. Let's try the following thought experiment: What is your favorite number? Tell me a random number? How many of you said "42"? How many chose a whole number? A positive number? A non-trivial fraction? A non-rational algebraic number? A transcendental number? While the reality is that almost all numbers are transcendental, we almost never pick those. Algebraic numbers have some extra aesthetic, being as the roots of polynomials.

It's reasonable to believe that when nature gives you an algebraic variety, it will have some symmetry. We need to learn how to exploit this symmetry to better understand the objects that are given to us. Geometric Complexity Theory (GCT) is no exception – symmetry is everywhere! The algebraic varieties that GCT gives us have natural symmetry, and being the good algebraic geometers that we are, we would like to understand the structure of their equations - their coordinate rings and their ideals, but instead of just using out of the box Gröbner basis techniques, which surely run out of steam even in the first couple of examples, we will work to exploit the natural symmetry that is handed to us by the problem.

In these lectures we will study the representation theory of GL_n, as this is the most natural symmetry group that arises in GCT. This is a classical subject, which one can learn about it in textbooks [FH91, Lan12, GW98]. I will try to give a streamlined approach that focuses on tensors, and symmetric tensors, and is especially amenable to computational methods.

Lecture 1: Algebraic Geometry and Commutative Algebra with Symmetry

2. Coordinate Rings: Symmetry Imposes its Will

Here's a motivating question: Suppose you are given an algebraic variety X, perhaps via a parametrization, you know that it is the zero-set of some polynomials, but *which ones*?

First consider projective space $\mathbb{P}V$, where we emphasize that it is the projectivization of a particular (finite dimensional) vector space V, that is $\mathbb{P}V$ is the space of all lines in V. The equivalence classes are $[x] \in \mathbb{P}V$ defined by $[x] = [y]$ iff $\exists \lambda \in \mathbb{C} \setminus \{0\}$ such that $x = \lambda y$. We simultaneously denote by $[x]$ the punctured line in V with direction vector x and the point in projective space. This equivalence is a symmetry of projective space, and since the punctured plane is topologically a circle, we often refer to it as a torus, denoted $\mathbb{T} = \mathbb{C} \setminus \{0\}$, and say that points of projective space are invariant under the torus action.

The coordinate ring of V is the polynomial ring $R = \mathbb{C}[V]$. The polynomial ring has a natural grading by degree, which one can see as a *consequence* of a torus action.

The torus invariance on projective space causes us to lose the meaning of *values* of polynomials on representatives of points in $\mathbb{P}V$, since we should have $f(x) = f(\lambda x)$ in order for the function f

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to be well-defined. Let us apply the $\mathbb{T}$-action to the mapping:

$$f\colon V \to \mathbb{C}$$

The $\mathbb{T}$-action decomposes V into a set of orbits, which come in two flavors. The origin $\vec{0}$ is the only 0-dimensional orbit, and rest are the lines $[v]$. So we can think of $V = \bigcup_{[v]\in\mathbb{P}V}[v] \cup \vec{0}$ as an orbit decomposition, where we emphasize the orbit $\vec{0}$ is contained in the closure of all the other orbits. Likewise $\mathbb{C} = [1] \cup 0$. Now we can view f as a mapping between $\mathbb{T}$-orbits:

$$f\colon V = \bigcup_{[v]\in\mathbb{P}V} [v] \cup \vec{0} \to \mathbb{C} = [1] \cup 0$$

If we want f to take the same value for every point on the same line, we see that we need the mapping to be compatible with the $\mathbb{T}$-action. As such, we see that the only thing that makes sense is whether the value of $f(x)$ is zero or non-zero.

We will re-define this notion later, but for now, given a group G and a vector space A, we will call A a G-module if it caries a G-action that is compatible with the vector space structure. If A and B are G-modules, we say that f is G-equivariant if $f(g.x) = g.f(x)$ for all $g \in G$ and all $x \in A$. [1] By construction G-equivariant maps take G-modules to G-modules. So we ask: what are the $\mathbb{T}$-equivariant polynomials?

Given a basis $\{x_0, \ldots, x_n\}$ of V we can write an element $f \in \mathbb{C}[V]$ as a *finite* linear combination of monomials: $f(x) = \sum_\alpha c_\alpha x^\alpha$, with multi-indices $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, coefficients $c_\alpha \in \mathbb{C}$, and monomials $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$.

Now consider what happens when we take a different points on the same line and ask under what conditions we can have equivariance, i.e. we need to solve for a constant μ in the equation $\mu f(x) = f(\lambda x) = \sum_\alpha c_\alpha \lambda^\alpha x^\alpha$ that works for all λ. We can convert this equation into a system of equations by noting that the monomials x^α form a basis of $\mathbb{C}[V]$:

$$\begin{array}{rcl}
\mu c_0 &=& c_0 \\
\mu c_{(1,0,\ldots 0)} &=& \lambda c_{(1,0,\ldots 0)} \\
&\vdots& \\
\mu c_{(0,0,\ldots 1)} &=& \lambda c_{(0,0,\ldots 1)} \\
\mu c_{(2,0,\ldots 0)} &=& \lambda^2 c_{(2,0,\ldots 0)} \\
\mu c_{(1,1,0\ldots 0)} &=& \lambda^2 c_{(1,1,0\ldots 0)} \\
&\vdots&
\end{array}$$

If f is the zero polynomial, then it sends everything to zero, an orbit, and hence is equivariant (and the parameter μ is free). If f is a non-zero constant polynomial then again it is equivariant since we can take $\mu = 1$.

If f is non-zero, then some equation $\mu c_\alpha = \lambda^d c_\alpha$ with $d = |\alpha| = \sum_i \alpha_i$, has $c_\alpha \neq 0$ and hence $\mu = \lambda^d$. If f has more than one term then we must have $\mu c_\alpha = \lambda^d c_\alpha$ and $\cdots$ and $\mu c_\beta = \lambda^e c_\beta$. By induction this is a contradiction unless $\lambda^e = \lambda^d$, and in particular $e = d$. Hence the only $\mathbb{T}$-equivariant polynomials have all nonzero terms having the same degree, i.e. the homogeneous polynomials. Note that the induced $\mathbb{T}$-action on degree d polynomials is given on monomials by $\lambda.(x^\alpha) = \lambda^\alpha x^\alpha$. I.e. it still acts by scalar multiplication, but it also records the degree. The elements for which $\mathbb{T}$ acts by the same scalar into $\mathbb{T}$-modules, denoted S^dV^*, the homogeneous polynomials of degree d on V. Note, we often forget the * for vector space dual, and for these lectures it causes little harm to assume $V^* = V$. We consider the zero function to have every degree so that S^dV is a vector space. Note that since the $\mathbb{T}$-action on V can be described without reference to basis ($v \mapsto \lambda v$) this definition is also basis-independent.

[1] Here we give the definition for both A and B left G-modules. One defines the actions with the inverse g^{-1} for right actions.

Exercise 2.1. Let $\mathbb{T}^{n+1}$ act on V (after a choice of basis) by re-scaling each variable. Find all $\mathbb{T}^{n+1}$-invariant polynomials.

Since equivalence classes of non-zero homogeneous polynomials are fixed by the $\mathbb{T}$-action we see that $\mathbb{C}[V]$ is a union of 1-dimensional $\mathbb{T}$-modules together with $\vec{0}$. This is a consequence of the fact that for abelian groups the only possible dimensions of irreducible modules are 0 and 1.

The degree function $\deg\colon \mathbb{C}[V] \to \mathbb{N}$ is defined by $\deg(x^\alpha) = |\alpha|$ on monomials, and $\deg(\sum_\alpha c_\alpha x^\alpha) = \max_{\alpha \mid c_\alpha \neq 0} |\alpha|$. Degree is finite because polynomials are finite linear combinations of monomials, and hence the maximum exists. Degree induces a filtration, $\mathbb{C}[V] = \bigcup S^{\leq d}V$, with $S^{\leq d}V$ denoting the polynomials with degree at most d.

One checks that degree is compatible with the ring structure. Namely that $\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$ for all $f, g \in S^{\leq d}V$, and $\deg(f \cdot g) = \deg(f) + \deg(g)$ for all $f, g \in S^{\leq d}V$ (since $\mathbb{C}$ has characteristic 0). We also obtain a *graded ring* structure on $\mathbb{C}[V] = \bigoplus_{d\in\mathbb{N}} S^dV$ since in addition $\deg(f+g) = d$ if $f, g \in S^dV$ (and $\deg(0) = d$).

Exercise 2.2. Describe an algorithm that a calculus student could come up with to take an element $f \in \mathbb{C}[V]$ and decompose it into is homogeneous pieces. That is, find a computational way to do the induction step: Given f find and extract all the highest degree piece.

A projective variety, denoted $X \subset \mathbb{P}V$, is an algebraic variety defined by the vanishing of *homogeneous* polynomials. Let $\mathcal{I}(X)$ denote the ideal of X, that is, the vector space of all polynomials that vanish on X.

Exercise 2.3. Prove that projective varieties are $\mathbb{T}$-invariant varieties and that a polynomial $f \in R = \mathbb{C}[V]$ vanishes on a projective variety X if and only if all its homogeneous pieces vanish. In particular the ideal of X inherits the grading and we obtain a graded R-module structure $\mathcal{I}(X) = \bigoplus_{d\in\mathbb{N}} \mathcal{I}(X)_d$, with $\mathcal{I}(X)_d := S^dV \cap I(X)$.

The grading on a commutative ring R invites a measurement defined for any finitely generated R-module $M = \bigoplus_{d\in\mathbb{N}} M_d$ called the Hilbert Function:

$$\begin{array}{rcl} \mathrm{HF}_M\colon \mathbb{N} & \to & \mathbb{N} \\ d & \mapsto & \dim M_d \end{array}$$

A key observation is that one can collect all these numbers into a series, the Hilbert series $\mathrm{HS}_M = \sum_{d\in\mathbb{N}} \mathrm{HF}_M(d) \cdot t^d$. This series has a rational representation

$$\mathrm{HS}_M = P_M(t)/Q_M(t),$$

and many important properties of M can be read off from the polynomials P and Q such as the algebraic degree of M, the Krull dimension of M, and the degrees of the generators of M.

Exercise 2.4. Show that if $\dim V = n+1$ then $\dim S^dV = \binom{n+d}{d}$, and find the rational representation of the Hilbert series for a polynomial ring on $n+1$ variables.

These objets enjoy Hilbert's namesake because of its connections to the three foundational theorems of Hilbert:

Hilbert's basis theorem: ideals over a polynomial ring on finitely many variables (or any Noetharian ring) are finitely generated.

Hilbert's Nullstellensatz: a bijective correspondence between affine algebraic varieties and prime ideals of polynomial rings.

Hilbert's syzygy theorem: minimal free resolutions of finitely generated R-modules exist and have finite length at most the number of variables.

Hilbert's work shocked the invariant theory world because he showed that for most of the groups G people cared about there exists a finite list of G-invariant polynomials, called fundamental invariants, that generate all other G-invariants. We'll see that with the help of computers we can give expressions for the Hilbert series for certain rings of invariants, and we can often use this to show that a given list of known fundamental invariants are complete.

Exercise 2.5. Show that computing any finite number of terms of the Hilbert series reduces to linear interpolation.

2.0.1. *Interpolation.* Consider what the grading gives us for the polynomial interpolation, which simply stated is this: given sufficiently many points x_i on an algebraic variety X, find a non-zero polynomial $f \in \mathbb{C}[V]$ that vanishes on all the x_i. If X is not known to be projective, we have to work with inhomogeneous polynomials and we have to go degree by degree computing the null space of the matrix

$$(m_\alpha(x_i))_{|\alpha| \leq s}$$

with m_α the monomial function that eats a vector x and produces the monomial x^α. The interpolation matrix has $\binom{n+s}{s}$ rows (by homogenization we can imagine the inhomogeneous polynomials as being homogeneous in one more variable and of the same degree).

If we knew that X were projective, we would do a similar null-space computation for each degree, but with $\binom{n-1+s}{s}$ rows. Here is an example of how quickly these numbers grow when $n = 4$. To find all polynomials that vanish on a variety for degree up to 10, one must construct square matrices of sizes: The complexity of the linear algebra required to compute a null space of an $n \times n$ matrix is essentially n^3 (but it is asymptotically ω, the complexity of matrix multiplication).

Already when $s = 10$ working with one more variable requires matrices with 3.5 as many rows (1001 versus 286), and 12.25 as many entries, and roughly 43 times as much computation. This storage and computational space can be the difference between being able to do one more computation or not.

2.1. **Algebraic varieties in GCT and their symmetry.**

2.1.1. *Bilinear complexity.* Consider a bilinear map on vector spaces $m\colon A \times B \to C$, such as matrix multiplication $m_{p,q,r}\colon \mathbb{C}^{p\times q} \times \mathbb{C}^{q\times r} \to \mathbb{C}^{p\times r}$. A fundamental question is how many multiplications over $\mathbb{C}$ must one do to compute m? A geometric way to state this is the following. It is a standard exercise to equate such bilinear maps to elements of the tensor space $A^* \otimes B^* \otimes C$. In order to treat each vector space on equal footing, we drop the *'s. The Segre product

$$\begin{array}{rcl} \mathrm{Seg}\colon \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C & \to & \mathbb{P}(A \otimes B \otimes C) \\ ([a],[b],[c]) & \mapsto & [a \otimes b \otimes c] \end{array}$$

parametrizes all bilinear maps that require exactly one multiplication to compute. These elements are also known as rank-1 elements, as indecomposable elements, or as separable states, depending on the context.

Given a variety $X \in \mathbb{P}V$ its k-th secant variety, denoted $\sigma_k(X) \subset \mathbb{P}V$, is the Zariski closure of the elements of X-rank k, i.e. those elements $[v] \in \mathbb{P}V$ such that $v = x_1 + \cdots + x_k$ with $[x_i] \in X$. Secant varieties form a chain $X = \sigma_1(X) \subset \sigma_2(X) \subset \cdots \sigma_k(X) \subset \cdots \subset \mathbb{P}V$ whose inclusions are strict until eventually it reaches the smallest projective space that contains X. We usually assume that X is linearly non-degenerate so that $\mathbb{P}V$ is the smallest dimensional projective space that contains X.

Given $m \in A \otimes B \otimes C$, the smallest r such that $[m] \in \sigma_r \operatorname{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is called the border rank of m, and it is the number of multiplications needed to compute any approximation of m.

Exercise 2.6. Show that X-border rank is semi-continuous, even though X-rank is not generally.

So we have a potential method for determining the approximation complexity of a bilinear mapping m - just find all the equations of all the secant varieties of triple Segre products, then test polynomial vanishing on m. Unfortunately this is a difficult proposition, and we don't know the generators of $\mathcal{I}(\sigma_k(\mathrm{Seg}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)))$ for $n \geq 4$.

Partial information can be gained, however. For instance lower bounds can be obtained by knowing any non-zero polynomial $f \in \mathcal{I}(\sigma_k(\mathrm{Seg}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)))$ then $f(m) \neq 0$ would provide a lower bound on complexity. For upper bounds one a complete set of set-theoretic defining equations would suffice, but this is a tall order.

One can start to see how difficult this problem would be to compute polynomials in n^3 variables to find the Hilbert function in low degree since this would involve interpolation with matrices of size $\binom{n^3+d-1}{d-1} \times \binom{n^3+d-1}{d-1}$. In order to make progress we need to understand the representation theory of $\mathrm{GL}_n^{\times 3}$. Examples of this sort of computation can be found [OS16, HIL13].

2.1.2. *VP vs VNP.* One major topic of this semester program is to understand the intersection between two orbit closures: The determinant and the padded permanent. One version of this is the following:

Consider $\det_n \in S^n\mathbb{C}^{n^2}$, and $\ell^{n-m}\operatorname{per}_m \in S^n\mathbb{C}^{m^2+1}$ respectively the determinant and padded (by ℓ a linear form) permanent families of polynomials. For what n, m do the orbit closures $\overline{\mathrm{GL}_{n^2}.\det_n}$ and $\overline{\mathrm{GL}_{n^2}.(\ell^{n-m}\operatorname{per}_m)}$ intersect? One approach is to compute the coordinate rings and compare them. These coordinate rings have a large group acting on them, which can help to reduce this computation, however still many interesting questions remain.

2.1.3. *Bigger tori.* Consider $\mathbb{T}^d$ and the action on d sets of variables thought of as the columns of an $n \times d$ matrix $(x_{i,j})$ by right multiplication by a diagonal matrix:

$$\begin{array}{rcl} \mathbb{T}^d \times V^{\oplus d} & \to & V^{\oplus d} \\ (\vec{\lambda}, (\vec{x}_1, \dots, \vec{x}_d)) & \to & (\lambda_1\vec{x}_1, \dots, \lambda_d\vec{x}_d) \end{array}$$

This action induces a grading on $\mathbb{C}[V^{\oplus d}]$ by multi-degree or weights. Let us set up notation to describe this. Let $e_1, \dots, e_d$ denote the standard basis of $\mathbb{Z}^d$. We say that the weight of a monomial is $\omega(x^\alpha) = \sum_i(\sum_j \alpha_{ij})e_j$. We say that an element of $S^e(V^{\oplus d})$ is a weight vector if all of its monomials have the same weight. One checks that weights are additive over products of weight vectors. We define the weight of an element only when its terms all have the same weight.

One still has the grading by total degree:

$$\mathbb{C}[V^{\oplus d}] = \bigoplus_{s \geq 0} \mathbb{C}[V^{\oplus d}]_s$$

The larger torus provides a finer grading:

$$\mathbb{C}[V^{\oplus d}]_s = \bigoplus_{|w|=s} \mathbb{C}[V^{\oplus d}]_{s,w}$$

where the individual pieces are the polynomials with total degree s and weight w.

Just as the single grading changed the search for homogeneous polynomials into a blocked linear algebra problem, this additional symmetry and it induced grading makes the blocks smaller.

As an example suppose one is looking for an expression of Cayley's $2 \times 2 \times 2$ hyperdeterminant via interpolation. It is known to be a degree 4 polynomial on the 8 variables of tensor space. The space of inhomogeneous polynomials of degree 4 has dimension 495, and the corresponding interpolation problem involves a matrix of size 495×495. Passing to homogeneous polynomials reduces to 330. However the number of *weight zero* monomials, i.e. those with the same number of 0's and 1's in each mode is only 12. This is a significant reduction. We can reduce even further, however, when we look at $\mathrm{SL}(2)$ invariance. In that case we will find that the space of $\mathrm{SL}(2)^{\times 3}$ invariants is only 2-dimensional, and the space of such invariants that lives in $S^4(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ is only 1-dimensional. So the interpolation problem drops from size 495 to 1 (just determining vanishing) by using symmetry.

2.1.4. *Fatter groups.* This is all motivation for why we want to know the fundamentals of representation theory.

2.1.5. *Representations of groups.* Suppose G is a group. We care mostly about compact groups, and finite groups, complex algebraic groups, and Lie groups, but usually we're thinking of G as either a symmetric group or a general linear group or special linear group. Classically groups were only studied when they acted on some space, and later their definition was abstracted to remove the necessary action on a space, but we can recover an action on a space by considering representations. Let V be a finite dimensional vector space over $\mathbb{C}$ and let $\mathrm{GL}(V)$ denote the invertible linear transformations $V \to V$.

A representation of a group G is a homomorphism

$$\rho\colon G \to \mathrm{GL}(V).$$

In such a case we say that V is a G-module, or that V carries a G-action. One can check that the definition of G-action is equivalent to having a representation. We often confuse terms and also call V a representation.

A sub-representation ρ' of a representation ρ of G is a vector subspace V' of V that is compatible with the same action on V given by ρ. We call V' a G-submodule of V. A sub-representation is called proper if it is not the zero vector space nor the entire space.

A representation is called irreducible if it does not have any proper submodules. The natural mappings between G-representations are G-module homomorphisms, which may also be called G-linear maps, or G-maps.

Noting the fact that the image, kernel and co-kernel of G-linear maps are G-modules one arrives at a key foundational result known as Schur's lemma, which characterizes maps between irreducible representations.

Lemma 2.7 (Schur's lemma)**.** *Suppose A and B are irreducible G-modules, and $\phi\colon A \to B$ a G-map. Either ϕ is an isomorphism or $\phi = 0$. If $A = B$ then $\phi = \lambda \cdot I$, a scalar times the identity for some $\lambda \in \mathbb{C}$.*

A group G is called completely reducible, reductive or semi-simple if it has the property that any (finite dimensional) representation of G is a direct sum of irreducible representations.

Proposition 2.8. *Suppose G is a finite group or a compact group. Then G is completely reducible.*

A standard proof of this fact is to first show for a finite dimensional representation every sub-representation has a G-invariant complement. This is accomplished in the finite case by averaging, and in the infinite compact case by integrating over the group.

A common theme in mathematics is to establish a so-called fundamental theorem. In arithmetic the fundamental theorem is that every integer can be factored as a product of prime powers and the factorization is unique up to changing the order. The classification of modules over a PID is similar. In representation theory the first half of our fundamental theorem is the previous proposition that says that for compact groups we can factor, and the next step for a given semi-simple group G be to identify all of its irreducible G-modules. The cases that interest us the most are when G is a symmetric group or a general linear group. We will study the symmetric group first, and then show how this work actually also answers the general linear group case.

2.2. **Representations of $\mathfrak{S}_d$, Young tableaux, Young symmetrizers, colored directed graphs.** The permutations on a finite set of size d is the group denoted $\mathfrak{S}_d$. In a first course on Algebra one encounters the following homework problems:

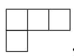
Exercise 2.9. (1) Show that conjugation $\sigma \mapsto x\sigma x^{-1}$ is an equivalence on $\mathfrak{S}_d$, and hence equivalence classes partition $\mathfrak{S}_d$.

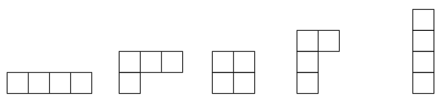
(2) Show that conjugation preserves cycle type, and hence these equivalence classes are determined by their cycle type.
(3) Show that disjoint cycles commute, so that cycle type does not depend on the order of the cycles, only the number of cycles of each length.

From these exercises one can deduce that the elements of $\mathfrak{S}_d$ are classified by cycle type, which in turn is indexed by partitions of d. It turns out that the irreducible representations of $\mathfrak{S}_d$ are indexed by the partitions of d.

We denote the set of partitions of d by $\{\pi \vdash d\}$. We list partitions in non-increasing order. So a partition may be denoted $\pi = (\pi_1, \dots, \pi_d)$, and we frequently drop the zero-parts in our notation, so the partition $(d, 0, \dots, 0)$ is denoted more compactly by (d). Likewise we indicate repetition in a partition by exponent, so $(1, \dots, 1) \vdash d$ is also denoted (1^d). The number of parts of a partition $\#\pi$ is the number of non-zero parts π_i and we define $|\cdot|$ by $|\pi| = \sum_i \pi_i$.

We won't prove this here, but we will show out to construct these representations and how to compute with them, i.e. we will show how to construct bases of these representations.

2.2.1. *Young diagrams, tableaux.* Partitions are depicted by Young diagrams, which we consider a diagram of boxes that are upper left justified. For example, the partition $(3, 1)$ is associated with the diagram , and the partitions of 4, for instance, are given with their diagrams below:



(4) $(3,1)$ $(2,2)$ $(2,1,1)$ $(1,1,1,1)$

A filing of a Young diagram (or Young tableau) is an assignment of a collection of (not necessarily distinct) letters or numbers (its content) to each of the boxes in the diagram. A tableau is said to be semi-standard if it is strictly increasing in the columns and weakly increasing in the rows. A tableau is said to be standard if it is strictly increasing in both rows and columns and each letter occurs once. The set of Standard Tableaux of shape π and content $\{1, \dots n\}$ is denoted $\mathrm{SYT}_\pi\{1, \dots, n\}$, and repetitions are allowed. For standard tableaux of shape $\pi \vdash$ the content is determined by the partition, so we drop it from the notation. For example, $\mathrm{SYT}_{(3,1)}$

$$\left\{ \begin{array}{ccc} 1&2&3\\ 4&& \end{array}, \begin{array}{ccc} 1&2&4\\ 3&& \end{array}, \begin{array}{ccc} 1&3&4\\ 2&& \end{array} \right\}.$$

We also note that the standard Young tableaux are in one-to-one correspondence with

Similarly we denote the set of Semi-Standard Tableaux by $\mathrm{SSYT}_\pi\{0, \dots, n\}$. Following are respectively all semi-standard tableaux of shape $(3, 1)$ and content from $\{0, 1, 2\}$: $\mathrm{SSYT}_{(3,1)}\{0, 1, 2\} =$

$$\left\{ \begin{array}{ccc} 0&0&0\\ 1&& \end{array}, \begin{array}{ccc} 0&0&0\\ 2&& \end{array}, \begin{array}{ccc} 0&0&1\\ 1&& \end{array}, \begin{array}{ccc} 0&0&1\\ 2&& \end{array}, \begin{array}{ccc} 0&0&2\\ 1&& \end{array}, \begin{array}{ccc} 0&0&2\\ 2&& \end{array}, \begin{array}{ccc} 0&1&1\\ 1&& \end{array}, \begin{array}{ccc} 0&1&1\\ 2&& \end{array}, \begin{array}{ccc} 0&1&2\\ 1&& \end{array}, \begin{array}{ccc} 0&1&2\\ 2&& \end{array}, \begin{array}{ccc} 0&2&2\\ 1&& \end{array}, \begin{array}{ccc} 0&2&2\\ 2&& \end{array}, \begin{array}{ccc} 1&1&1\\ 2&& \end{array}, \begin{array}{ccc} 1&1&2\\ 2&& \end{array}, \begin{array}{ccc} 1&2&2\\ 2&& \end{array} \right\}$$

Note that in `Macaulay2` one can use the package `PieriMaps` and the command `standardTableaux` to produce sets of semi-standard tableaux.

There are combinatorial formulas, known as hook-length formulas for counting the numbers of semi-standard and standard tableaux.

To each box in a Young diagram we assign the hook length by counting that box together with the number of boxes directly to the right (the arm) and directly below (the leg). If the pair (i, j) denotes the location of a box in row i and column j in Y_π write $(i, j) \in Y_\pi$ and let $h_{i,j}$ denote the hook length of the hook cornered at box (i, j). It is convenient to record the hook lengths in a filling of the tableaux. For example, the hook lengths of the Young diagram of shape $(4, 3, 2, 1)$ are

$$\begin{array}{cccc} 7&5&3&1\\ 5&3&1&\\ 3&1&&\\ 1&&& \end{array}.$$

The number of semi-standard tableaux of shape $\pi \vdash d$ filled with numbers $\{0, 1, \dots, n\}$ is given by

$$\#\,\mathrm{SSYT}_\pi\{0,\dots,n\} = \prod_{(i,j)\in Y_\pi} \frac{n+1+j-i}{h_{i,j}}. \tag{1}$$

It is convenient to write this dimension as a ratio of tableaux, taking the product of the contents of each of the tableaux. For instance,

$$\#\,\mathrm{SSYT}_{(3,2,1)}\{0,1,2\} = \begin{ytableau} 3 & 4 & 5 \\ 2 & 3 \\ 1 \end{ytableau} \Big/ \begin{ytableau} 5 & 3 & 1 \\ 3 & 1 \\ 1 \end{ytableau} = 8.$$

Similarly, the number of standard tableaux of shape $\pi \vdash d$ is given by a hook-length formula:

$$\#\,\mathrm{SYT}_\pi = \frac{d!}{\prod_{(i,j)\in Y_\pi} h_{i,j}}. \tag{2}$$

Again using ratio of tableaux, taking the product of the contents of each of the tableaux we have for $(3,2,1) \vdash 6 = d$:

$$\#\,\mathrm{SYT}_{(3,2,1)} = \begin{ytableau} 6 & 5 & 4 & 3 & 2 & 1 \end{ytableau} \Big/ \begin{ytableau} 5 & 3 & 1 \\ 3 & 1 \\ 1 \end{ytableau} = 16,$$

and for $(3,1) \vdash 4 = d$:

$$\#\,\mathrm{SYT}_{(3,1)} = \begin{ytableau} 4 & 3 & 2 & 1 \end{ytableau} \Big/ \begin{ytableau} 4 & 2 & 1 \\ 1 \end{ytableau} = 3.$$

Exercise 2.10. Show that for the staircase diagram $\pi = (n, n-1, \dots, 1, 0)$ we have $\#\,\mathrm{SSYT}_\pi\{0,\dots n\} = 2^{\binom{n+1}{2}}$.

We will show in the following sections the computational aspects of the following facts that we now take for granted:

(1) The dimension of the irreducible $\mathfrak{S}_d$ module indexed by π is equal to $\#\,\mathrm{SYT}_\pi$, and hence we can use the standard Young tableaux as a basis of an abstract vector space of the same dimension.
(2) The dimension of the irreducible $\mathrm{SL}(V)$ module indexed by π, with $V \cong \mathbb{C}^{n+1}$, is equal to $\#\,\mathrm{SSYT}_\pi\{0,\dots,n\}$, and hence we can use the semi-standard Young tableaux as a basis of an abstract vector space of the same dimension. We denote this irreducible module using the Schur functor notation: $S_\pi V$.

2.2.2. *Young symmetrizers.* Young symmetrizers associated with a partition $\pi \vdash d$ depend on the choice of a filling T_π, and are constructed as operators c_π (typically denoted without reference to the filling) on $V^{\otimes d}$, the d-th tensor power of a vector space V. Typically one does this in two steps, a skew-symmetrization b_π and a symmetrization a_π, and the tableau gives the recipe for how to construct these maps. Further, one notes that tableaux of the same shape produce isomorphic, but not-necessarily equal, images. The equivalence class of Young symmetrizers associated with shape π is an irreducible representation of $\mathfrak{S}_d$. Note that non-zero scalar multiples of linear mappings produce equivalent mappings, so it makes sense to look at the group algebra $\mathbb{C}\mathfrak{S}_d$ formed by formal complex linear combinations of permutations, and then consider the image of one such c_π under the action of the group algebra. This is the $\mathbb{C}$-span of the orbit of $\mathfrak{S}_d$ of a particular c_π. It's clear that this is spanned by the Young symmetrizers associated with all fillings (with no repetitions), but what is less clear, but is perhaps evident from the straightening laws on tableaux, is that the symmetrizers associated with standard fillings form a basis of this space.

The image in $V^{\otimes d}$ of a Young symmetrizer c_π is (a copy of) an irreducible $\mathrm{SL}(V)$ module, the Schur module $S_\pi V$, and in fact all such irreps of $SL(V)$ arise in this way.

Let us elaborate. Let $\bigwedge^k V$ and $S^k V$ respectively denote the alternating and symmetric powers of V inside $V^{\otimes k}$. They are respectively the images of the Young symmetrizers associated with partitions (1^k) and (k). The Young symmetrizer $c_{(1^k)}$ consists of only the skew-symmetrization map $b_{(1^k)}$, the symmetrization map $a_{(1^k)}$ being trivial. The reverse is true for the symmetrizer $c_{(k)}$. Specifically, we describe the linear mapping (it's actually a projection) on a basis of indecomposable elements as follows:

$$\begin{aligned} c_{(1^k)} = b_{(1^k)} \colon V^{\otimes k} &\to \textstyle\bigwedge^k V \\ v_1 \otimes \cdots \otimes v_k &\mapsto v_1 \wedge \cdots \wedge v_k, \end{aligned}$$

and

$$\begin{aligned} c_{(k)} = a_{(k)} \colon V^{\otimes k} &\to S^k V \\ v_1 \otimes \cdots \otimes v_k &\mapsto v_1 \circ \cdots \circ v_k. \end{aligned}$$

One checks that each of these mappings is in fact a projection by using the wedge $\wedge$ and symmetric $\circ$ products of elements in the tensor product space:

$$\begin{aligned} v_1 \wedge \cdots \wedge v_k &:= \tfrac{1}{k!} \textstyle\sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}, \\ v_1 \circ \cdots \circ v_k &:= \tfrac{1}{k!} \textstyle\sum_{\sigma \in \mathfrak{S}_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}. \end{aligned} \tag{3}$$

Note that since there is only one standard filling of each of the tableaux (1^k) and (k) there is no ambiguity in defining $c_{(1^k)}$ and $c_{(k)}$.

Now, for concreteness, we arbitrarily decide to work with the standard filling of a tableau that fills by columns first, and we might be tempted to call this the canonical filling, but this is an abuse of language because there is no reason this choice should be, in fact, canonical, however we will see that any choice will do. When working with multiple tensor factors we will need greater flexibility and this requirement will slightly overcomplicate what follows.

The recipe for producing the Young symmetrizer, denoted c_{T_π} associated with a filling T_π of shape $\pi \vdash d$ is as follows. Set $p := \#\pi$. Let $\lambda = \pi'$ be the transpose partition whose entries count the number of boxes in each column of π, and let $\ell := \#\lambda$. Construct b_{T_π} and a_{T_π} by the following. Add indices $1, \dots d$ to d distinct copies of V, and denote them V_i, and let $\bigwedge^k V_I$, respectively $S^d V_I$, with $|I| = k$ denote the skew-symmetrization (respectively symmetrization) of the copies indexed by I. Let $R_i T_\pi$ denote the content of row i of T_π and similarly let $C_i T_\pi$ denote the content of column i.

$$b_{T_\pi} \colon V_1 \otimes \cdots \otimes V_d \to \textstyle\bigwedge^{\lambda_1} V_{C_1 T_\pi} \otimes \bigwedge^{\lambda_2} V_{C_2 T_\pi} \otimes \cdots \otimes \bigwedge^{\lambda_\ell} V_{C_\ell T_\pi}$$

and

$$a_{T_\pi} \colon V_1 \otimes \cdots \otimes V_d \to S\pi_1 V_{R_1 T_\pi} \otimes S\pi_2 V_{R_2 T_\pi} \otimes \cdots \otimes S\pi_p V_{R_p T_\pi}$$

Erasing the indices for each gives tensor products of exterior, respectively symmetric, powers of V, but it forgets which copies of V were used in that construction. This information is important for the construction of irreducible representations of products of several $\mathrm{SL}(V)$'s. At any rate, the concise description of the images is the same for any filling so we drop the reference to the filling and write

$$\operatorname{Im}(b_\pi) \cong \textstyle\bigwedge^{\lambda_1} V \otimes \bigwedge^{\lambda_2} V \otimes \cdots \otimes \bigwedge^{\lambda_\ell} V \subset V^{\otimes d}$$

and respectively

$$\operatorname{Im}(a_\pi) \cong S\pi_1 V \otimes S\pi_2 V \otimes \cdots \otimes S\pi_p V \subset V^{\otimes d}.$$

Note that (3) implies that the images of these maps are subspaces of $V^{\otimes d}$ so that they can be composed to form the the Young symmetrizer $c_\pi = a_\pi \circ b_\pi$. The image of c_π is (a copy of) the irreducible $\mathrm{SL}(V)$-representation, which is called a Schur module, is denoted $S_\pi V \subset V^{\otimes d}$. In addition, the claim from representation theory is that the equivalence class $\mathbb{C}\mathfrak{S}_d c_{T_\pi}$ is an irreducible $\mathfrak{S}_d$-module. This module, which is often called a Spect module, is sometimes denoted $[\pi]$. As mentioned above, we can take the standard Young tableaux to be a formal basis for $[\pi]$ but this discussion shows how one obtains elements in the space of linear operators $V^{\otimes d} \to V^{\otimes d}$ to realize these elements. However, it is clear that not every tableau produces a distinct mapping - at least one ambiguity can be seen by re-ordering columns that have the same height - this operation fixes both b_π and a_π. The complete set of such rules is called a straightening law, which allows one to take any tableau and move it to a linear combination of standard tableaux. We don't have time to discuss this in detail. The claim that we're not going to prove is that one can use a straightening law to show that the operators associated with standard Tableaux form a basis of the space spanned by all Young symmetrizers associated with shape π (i.e. for any filling, not just the standard ones). We summarize this (without proof) as follows.

Theorem 2.11. *The irreducible $\mathfrak{S}_d$-modules are all of the form $[pi] = \mathbb{C}\mathfrak{S}_d.c_{T_\pi}$, with T_π any filling of shape π, in particular, $[\pi]$ is spanned by the set of Young symmetrizers as the fillings of shape π run over all possibilities. Moreover $[\pi]$ has a basis $\{c_{T_\pi} \mid T_\pi \in \mathrm{SYT}_\pi\}$.*

2.3. **Weight bases of Schur modules.** In what follows we will construct bases of Schur modules utilizing Young symmetrizers. Moreover, these bases respect the wights induced by the natural torus action $T^n \subset \mathrm{SL}(V)$. For ease of notation, however, we choose to expand our scope just slightly to consider the natural torus action $T^{n+1} \subset \mathrm{GL}(V)$, and we view $\mathrm{GL}(V)$ as being isomorphic to the product $\mathrm{SL}(V) \times \mathbb{T}^1$, where in the later case the torus is multiplication by the determinant, specifically, the isomorphism is $A \mapsto (A/\det(A), \det(A))$.

The choice of a basis of V that is compatible with the torus action is essentially equivalent to choosing a torus in $\mathrm{GL}(V)$. We denote this basis $x_0, \ldots, x_n$ and we insist that $\mathbb{T}^{n+1}$ acts by re-scaling these vectors, i.e. $\mathbb{T}^{n+1}$ is the diagonal matrices in GL_{n+1}. Let e_i denote a basis of $\mathbb{Z}^{n+1}$. We define a weight function ω by a multi-step process as follows. Set $\omega(x_i) = e_i$. Later, we may want to define the dual vector space, denoted V^* of linear operators to the ground field, with corresponding dual basis denoted x^i, with correspondence $x^j(x_i) = \delta_{i,j}$, the Kronecker delta. In that case we also have weight $\omega(x^i) = -e_i$. We declare that weights are additive over tensor products (and hence over wedge and symmetric products as well). We say that an element of $V^{\otimes d}$ is a weight vector if every term in an expansion in the standard basis has the same weight.

This definition of weight is compatible with the induced action of $\mathrm{SL}(V)$ and $\mathrm{GL}(V)$ on $V^{\otimes}d$. Moreover, it is straightforward to check that the Young symmetrizers preserve weight, and hence are $\mathbb{T}^{n+1}$-equivariant. Because of this we can obtain a weight basis of the Schur module $S_\pi V$ by feeding a Young Symmetrizer an *appropriate* set of weight vectors. The magic here is that "appropriate" is entirely captured by the notion of Semi-Standard Young Tableaux. Let T_π be a tableau of shape π with content exactly $\{1, \ldots, d\}$ (no repetitions). Then just as T_π gives a recipe for constructing (a copy of) the symmetrizer c_π, it also gives a way to assign a tensor product of basis vectors to any filling of a tableau, which we call the reading order. Given any filling Y_π with content from $\{0, \ldots, n\}$ replace box $\boxed{i}$ with vector x_i and send Y_π to the tensor product of its content placed in order given by the reading order.

For example, given $T_{(3,1)} = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}$, one associates to $Y_{(3,1)} = \begin{array}{|c|c|c|}\hline 0&2&2\\\hline 1\\\cline{1-1}\end{array}$ the tensor $x_0 \otimes x_2 \otimes x_1 \otimes x_2$, which has weight $e_0 + e_2 + e_1 + e_2 = (1, 1, 2)$.

The content of a tableau is the same information as the weight of the associated tensor, and the reading tableau does not change the weight.

Theorem 2.12. *The set of tableaux of shape π form a spanning set for the Schur module $S_\pi V$, and $\mathrm{SSYT}_\pi\{0, \ldots, n\}$ forms a weight basis of $S_\pi V$.*

One can put a partial order on the set of weights, and then declare a highest weight that occurs with non-zero coefficient in $S_\pi V$. With the standard lex partial order on $\mathbb{Z}^{n+1}$ the highest weight is associated with what one may call the canonical filling of shape π, which is the filling that has π_i $\boxed{i}$'s in the i-th row for all i. Hence the highest weight for $S_\pi V$ is π, thought of as an integer vector in $\mathbb{Z}^{n+1}$.

Remark 2.13. As a caution, this notion of weight, which could be thought of as a tensor multi-degree, does not match with the common notion of fundamental weights. For $\mathrm{SL}(V)$ the fundamental representations are the wedge powers $\bigwedge^k V$, and the highest weight of such representation is denoted ω_k. The irreducible representation with highest weight $a_1\omega_1 + \cdots a_n\omega_n$ is denoted $\Gamma_{a_1,\dots a_n}$. By comparing weights, the representation Γ_λ must be the Schur module $S_\pi V$ with $\pi = \lambda'$.

Remark 2.14. A word on duality. One can check that the duality of vector spaces induces the following isomorphisms of $\mathrm{SL}(V)$-modules $(S_\pi V)^* \cong S_\pi V^* \cong S_{\pi^*} V$, where π^* is the contragradient partition, which can be formed by taking a $(n+1) \times \#\pi$ box, placing π in the upper left corner and removing those boxes, then take the remaining boxes and rotate 180 degrees. Equivalently $\pi^* = ((n+1)^{\#\pi}) - \overleftarrow{\pi}$, with $\overleftarrow{\pi} := (\pi_{\#\pi}, \dots, \pi_1)$, the reverse. On the level of fundamental weights, the contragradient reverses the order up to a twist by the trivial weight.

2.3.1. *Representations of* $\mathrm{GL}(V)$ *from* $\mathrm{SL}(V)$. Since $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$ differ by a torus $\mathbb{T}^1$ their representations are essentially the same, up to a twist by a trivial (1-dimensional) representation. We can imagine the trivial representation $\mathbb{C}$ as a Schur module such as $\bigwedge^{n+1} V$, which is also 1-dimensional. This representation is a preferable choice since the extra torus action is multiplication by the determinant, and that is how $\mathrm{GL}(V)$ acts on $\bigwedge^{n+1} V$. However, since all representations of a torus (which is abelian) are 1-dimensional, the inverse $(\bigwedge^{n+1} V)^{-1}$ also makes sense, and from the point of view of the contragradient we have a vector space isomorphism $\bigwedge^k V^* \cong \bigwedge^{n+1-k} V^*$, however the degree of this isomorphism is wrong. So we twist it and get

$$\bigwedge^k V^* \cong \bigwedge^{n+1-k} V \otimes \bigwedge^{n+1} V^*.$$

So, we can take any integer power of the trivial representation $(\bigwedge^{n+1} V)^t$ and get a different copy of the trivial representation for each integer s, which we call the degree. So we arrive at the following (which we don't prove here):

Theorem 2.15. *For $V \cong \mathbb{C}^{n+1}$, all the irreducible representations of* $\mathrm{GL}(V)$ *are of the form*

$$S_\pi V(-t) \cong S_\pi V \otimes (\bigwedge^{n+1} V)^t,$$

with $\pi \vdash d$ and $t \in \mathbb{Z}$.

2.3.2. *Schur - Weyl Duality, the double commutant theorem.* We have already seen that the representations of $\mathfrak{S}_d$ (Spect modules) and representations of $\mathrm{GL}(V)$ (Schur modules) are connected (via Young symmetrizers). This can be made more formal. Note that $V^{\otimes d}$ has an action of $\mathfrak{S}_d$ (permuting the factors) as well an action of $\mathrm{GL}(V)$ (simultaneous change of coordinates in each factor). Hence both groups can be embedded in $\mathrm{GL}(V^{\otimes d})$. It's easy to see that the two actions commute, what's more is that they are actually the commutants of each other, due to theorem of Schur. A consequence of this is that we can consider the joint action of $\mathfrak{S}_d \times \mathrm{GL}(V)$ on $V^{\otimes d}$, and obtain the following decomposition.

Theorem 2.16 (Schur-Weyl duality)**.** *As $\mathfrak{S}_d \times \mathrm{GL}(V)$-modules*

$$V^{\otimes d} = \bigoplus_{\pi \vdash d} [\pi] \otimes S_\pi V. \tag{4}$$

We note that this decomposition is multiplicity-free, and that the Spect module $[\pi]$ uniquely determines the Schur module, and vice versa. So this statement also includes a 1-1 correspondence between the representations of $\mathfrak{S}_d$ and $\mathrm{SL}(V)$.

Exercise 2.17. Set $n = 2$ and $d = 3$. Check that the dimensions of the representations on both sides of the equation in the theorem add up.

2.4. **Construction of irreducible** $\mathrm{SL}(V_1)\times\cdots\times\mathrm{SL}(V_d)$ **modules.** Suppose $V_1, \dots V_d$ are $\mathbb{C}$-vector spaces of respective dimensions $n_1, \dots, n_d$ (all finite).

Proposition 2.18 (4.2.5 [GW98])**.** *Suppose G_1 and G_2 are reductive linear algebraic groups. The irreducible representations of $G_1 \times G_2$ are all of the form $E_1 \otimes E_2$ for E_1 and E_2 respectively irreducible representations of G_1 and G_2.*

From this we immediately obtain the following

Proposition 2.19. *The irreducible representations of $\mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_d)$ are all of the form*

$$S_{\vec{\pi}}\vec{V} := S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_d}V_d.$$

The notation $\vec{\pi}$ is to indicate a multi-partition $\vec{\pi} = (\pi_1, \dots, \pi_d)$ with each π_i itself a partition of a non-negative integer $|\pi_i|$. So as an abuse of notation, before π_i was a part of a partition, now it is a partition itself.

Since the difference between $\mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_d)$ and $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_d)$ is the torus $\mathbb{T}^d$, so we also obtain the same representations up to a twist:

Proposition 2.20. *The irreducible representations of $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_d)$ are all of the form*

$$S_{\vec{\pi}}\vec{V}(-\vec{t}\,) := S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_d}V_d \otimes \left((\textstyle\bigwedge^{n_1}V_1)^{t_1} \otimes \cdots \otimes (\textstyle\bigwedge^{n_d}V_d)^{t_d}\right).$$

Note that each $S_{\vec{\pi}}\vec{V}(-\vec{t}\,)$ is a sub-representation of $V^{\otimes d}$, and is constructed simply by the tensor product of Young symmetrizers.

2.5. **Decomposition and computation for irreducible** $\mathrm{SL}(V)$ **modules inside** $\mathbb{C}[V_1\otimes\cdots\otimes V_d]$**.** Now that we know the representations of $\mathrm{SL}(V)$, $\mathrm{GL}(V)$ and products of such we can finally begin to describe the representations that occur in the coordinate ring of a tensor product $\mathbb{C}[V_1\otimes\cdots\otimes V_d]$. As before, we have grading by degree (and we omit *'s for duals at this point for notational simplicity):

$$\mathbb{C}[V_1 \otimes \cdots \otimes V_d] = \bigoplus_{e\geq 0} S^e(V_1 \otimes \cdots \otimes V_d)$$

The following classical formulas are called the Cauchy formulas, and tell how to decompose symmetric and exterior powers of sums and tensor products of vector spaces V_1 and V_2 as $\mathrm{SL}(V_1)\times\mathrm{SL}(V_2)$-modules.

Proposition 2.21 (Cauchy Formulas)**.**

$$\begin{aligned} S^d(V_1 \oplus V_2) &= \textstyle\bigoplus_{e=0}^d S^{d-e}V_1 \otimes S^eV_2, & S^d(V_1 \otimes V_2) &= \textstyle\bigoplus_{\pi\vdash d} S_\pi V_1 \otimes S_\pi V_2, \\ \textstyle\bigwedge^d(V_1 \oplus V_2) &= \textstyle\bigoplus_{e=0}^d \bigwedge^{d-e}V_1 \otimes \bigwedge^e V_2, & \textstyle\bigwedge^d(V_1 \otimes V_2) &= \textstyle\bigoplus_{\pi\vdash d} S_\pi V_1 \otimes S_{\pi'} V_2. \end{aligned} \tag{5}$$

Attempting to apply the Cauchy formulas to the coordinate ring of a tensor product we see that we also need to consider compositions of Schur functors. This can be done via character theory for small partitions and for special cases, but in general these problems can be just as hard as the complexity theory questions of relevance for this semester. So for now we just state the result and then we will explain a method for computing bases of these modules which would imply the character theory results, but is arguably not the best way to compute characters.

Proposition 2.22 (Landsberg-Manivel, [Lan12, Prop. 6.5.1.2])**.**

$$\mathbb{C}[V_1 \otimes \cdots \otimes V_d] = \bigoplus_{e\geq 0}\bigoplus_{\vec{\pi}\vdash e} S_{\vec{\pi}}\vec{V}^{\oplus N_{\vec{\pi}}}, \tag{6}$$

with $\vec{\pi} \vdash e$ a multi-partition with partitions $\pi_i \vdash e$ for $1 \leq i \leq d$, and the multiplicity $N_{\vec{\pi}}$ is the dimension of the space of $\mathfrak{S}_d$-invariants $([\pi_1] \otimes \cdots \otimes [\pi_d])^{\mathfrak{S}}_d$.

This decomposition is an example of an *isotypic* decomposition, and the numbers $N_{\vec{\pi}}$ are known as *multiplicities*. In the case $d = 2$ we may use the Cauchy formulas to compute $N_{\vec{\pi}}$, however, when $d = 3$ the $N_{\vec{\pi}}$ are the Kronecker coefficients. And for $d > 3$ the $N_{\vec{\pi}}$ can be computed from the Kronecker coefficients.

We may also consider the multiplicity as a vector space with a module structure, which is known as the highest weight space:

$$S_{\vec{\pi}}\vec{V}^{\oplus N_{\vec{\pi}}} = S_{\vec{\pi}}\vec{V} \otimes M_{\vec{\pi}}$$

with $M_{\vec{\pi}}$ a $\mathfrak{S}_d$ module with dimension $N_{\vec{\pi}}$. An advantage of this approach is that one may attempt to find a basis of $M_{\vec{\pi}}$, which also will compute the number $N_{\vec{\pi}}$. In addition, one may compute may be able to compute a non-zero vector in $M_{\vec{\pi}}$ without even knowing $N_{\vec{\pi}}$.

2.5.1. *Weaving.* One approach to the following is *bird tracks* which have their advantage as being quite visual, but the disadvantage is that they may require a lot of typesetting for notes. We choose to continue with the non-pictorial exposition.

Our starting observation is that up to isomorphism tensor product is commutative on the level of vector spaces, so $V_1 \otimes V_2 \cong V_2 \otimes V_1$. We're interested in computing irreducible representations in $S^e(\vec{V})$, which is the $\mathfrak{S}_d$-invariant space inside $(\vec{V})^{\otimes e}$. When we expand the tensor power we get:

$$(\vec{V})^{\otimes e} = \vec{V} \otimes \vec{V} \otimes \cdots \vec{V} = (V_1 \otimes \cdots V_d) \otimes \cdots \otimes (V_1 \otimes \cdots V_d)$$

We notice that the vector spaces are not in the correct order to apply Schur-Weyl duality, but by the commutativity on the level of vector spaces we can weave the vector spaces to obtain

$$(\vec{V})^{\otimes e} \cong (V_1)^{\otimes e} \otimes \cdots \otimes (V_d)^{\otimes e}.$$

Now one can apply Schur-Weyl duality (4) to each vector space to obtain

$$(\vec{V})^{\otimes e} \cong \bigoplus_{\pi_1 \vdash e} [\pi_1] \otimes S_{\pi_1}V_1 \otimes \cdots \otimes \bigoplus_{\pi_d \vdash e} [\pi_d] \otimes S_{\pi_d}V_d.$$

Then collecting the $\mathfrak{S}_d$ modules we obtain

$$(\vec{V})^{\otimes e} \cong \bigoplus_{\vec{\pi} \vdash e} ([\pi_1] \otimes \cdots \otimes [\pi_d]) \otimes S_{\vec{\pi}}\vec{V}.$$

The decomposition of $S^e(\vec{V})$ arises by taking $\mathfrak{S}_e$ invariants. The above discussion is Landsberg and Manivel's proof of Proposition 2.22. However, we can turn the last step into a computational method by computing the projection operator on the tensor product of Young symmetrizers. We can do this (for small enough partitions) and simultaneously obtain a basis of the highest weight space of the $S_{\vec{\pi}}\vec{V}$. The most naive way to compute the projection onto $\mathfrak{S}_e$-invariants is to apply the averaging operator, which is, for any finite group G acting on $v \in V$ is $\Omega_G(v) = \frac{1}{|G|}\sum_{g \in G} g.v$. One additional trick we can employ is that the averaging operator in our situation is essentially changing $\otimes$-products for $\circ$-products. Symmetrizing $v_1 \otimes \cdots \otimes v_n$ produces $v_1 \circ \cdots \circ v_n$. The challenge is to be able to do this in the weaved setting. This is perhaps best seen by example.

2.5.2. *degree 4 binary invariants.* There is a 1-dimensional space of $\mathrm{GL}(2)^{\times 3}$ invariants in $\mathbb{C}[\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2]$ in degree 4, which could be computed by character theory. Let us see how to discover this from Young symmetrizers.

The tableau associated to the partition $(2, 2)$ has 2 standard fillings:

$$T_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \; T_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

This means that the representation $S_{2,2}\mathbb{C}^2$ occurs with multiplicity 2 inside $(\mathbb{C}^2)^{\otimes 4}$. Quickly one sees that it is important to distinguish different copies of $\mathbb{C}^2$, so we set $A = B = C = \mathbb{C}^2$.

The space $S_{2,2}A \otimes S_{2,2}B \otimes S_{2,2}C$ occurs with multiplicity 2^3 inside $(A \otimes B \otimes C)^{\otimes 4}$, and the multiplicity space has basis $\{T_i \otimes T_j \otimes T_k \mid i,j,k \in \{1,2\}$. Because globally re-naming the content of tableau is just an action of $\mathfrak{S}_4$, when computing the projection to the space of invariants we only need to consider 2 types: $T_1 \otimes T_1 \otimes T_1$ and $T_1 \otimes T_1 \otimes T_2$. Applying shuffling rules like in [Rai12, OR14] one could find out that only the second of these projects to a nonzero invariant as the first is simultaneously symmetric and skew-symmetric and must project to zero. But we can compute it directly as well. We present an algorithm that we have used several times, see [BO11, OS16] for instance. This algorithm has the feature that it can be used to produce the expression of the polynomial in the image and it can be used to evaluate this polynomial without first finding its expression (which is much more efficient).

First compute C_{T_1} on A, which is the mapping

$$A^{\otimes 4} \xrightarrow{b_{T_1}} \bigwedge^2 A_{1,2} \otimes \bigwedge^2 A_{3,4} \xrightarrow{\iota} A^{\otimes 4} \xrightarrow{a_{T_1}} S^2 A_{1,3} \otimes S^2 A_{2,4} \xrightarrow{\iota} A^{\otimes 4}.$$

Evaluate this on a simple tensor $a_1 \otimes a_2 \otimes a_3 \otimes a_4$:

$$a_1 \otimes a_2 \otimes a_3 \otimes a_4 \mapsto (a_1 \wedge a_2) \otimes (a_3 \wedge a_4) = (a_{1,1} \otimes a_{2,2} - a_{1,2} \otimes a_{2,1}) \otimes (a_{3,1} \otimes a_{4,2} - a_{3,2} \otimes a_{4,1}).$$

We can also view this action as eating a filled tableau $T = \begin{array}{|c|c|}\hline \mathrm{a}_1 & \mathrm{a}_3 \\ \hline \mathrm{a}_2 & \mathrm{a}_4 \\ \hline \end{array}$ and skew-symmetrizing its columns. The standard tableau $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$ tells the order in which to read things to make a tensor in $A^{\otimes 4}$, which in this case puts the content of the filling in order as a tensor.

Computationally if we have an index that unambiguously records the order of terms one can use the usual (commutative) product in place of the tensor product with no loss of information. So we can view the result as

$$\begin{aligned} b_{T_1}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) &= (a_{1,1}a_{2,2} - a_{1,2}a_{2,1})(a_{3,1}a_{4,2} - a_{3,2}a_{4,1}) \\ &= a_{1,2}a_{2,1}a_{3,2}a_{4,1} - a_{1,1}a_{2,2}a_{3,2}a_{4,1} - a_{1,2}a_{2,1}a_{3,1}a_{4,2} + a_{1,1}a_{2,2}a_{3,1}a_{4,2} \end{aligned}$$

which is an element of $\bigwedge^2 A_{1,2} \otimes \bigwedge^2 A_{3,4}$. It is no coincidence that the expression of b_{T_π} is a product of determinants. The general procedure is to take the vectors to be skew-symmetrized and put them in the *rows* of a matrix. The reason for this convention is our tradition of appending indices to the right when writing the entries of a vector that already has a subscript, for instance $a_1 = (a_{1,1}, \ldots, a_{1,n})$ looks like a row vector. So we can also write

$$b_{T_1}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = \begin{vmatrix} \mathrm{a}_{1,1} & \mathrm{a}_{1,2} \\ \mathrm{a}_{2,1} & \mathrm{a}_{2,2} \end{vmatrix} \begin{vmatrix} \mathrm{a}_{3,1} & \mathrm{a}_{3,2} \\ \mathrm{a}_{4,1} & \mathrm{a}_{4,2} \end{vmatrix}.$$

Similarly,

$$b_{T_2}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = \begin{vmatrix} \mathrm{a}_{1,1} & \mathrm{a}_{1,2} \\ \mathrm{a}_{3,1} & \mathrm{a}_{3,2} \end{vmatrix} \begin{vmatrix} \mathrm{a}_{2,1} & \mathrm{a}_{2,2} \\ \mathrm{a}_{4,1} & \mathrm{a}_{4,2} \end{vmatrix}.$$

Next, we weave the terms so that the indices we want to symmetrize over are adjacent.

$$b_{T_1}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = a_{1,2}a_{3,2}a_{2,1}a_{4,1} - a_{1,1}a_{3,2}a_{2,2}a_{4,1} - a_{1,2}a_{3,1}a_{2,1}a_{4,2} + a_{1,1}a_{3,1}a_{2,2}a_{4,2}$$

Applying the symmetrization produces:

$$a_{T_1} \circ b_{T_1}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = a_{1,2} \circ a_{3,2} \otimes a_{2,1} \circ a_{4,1} - a_{1,1} \circ a_{3,2} \otimes a_{2,2} \circ a_{4,1} - a_{1,2} \circ a_{3,1} \otimes a_{2,1} \circ a_{4,2} + a_{1,1} \circ a_{3,1} \otimes a_{2,2} \circ a_{4,2}$$

Now one could change notation to indicate that the variables with first index 1 or 3 are in the same space, and similarly that variables with first index 2 or 4 are in the same space.

$$\begin{gathered} a_{T_1} \circ b_{T_1}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = \\ a_{13,2}a_{13,2}a_{24,1}a_{24,1} - a_{13,1}a_{13,2}a_{24,2}a_{24,1} - a_{13,2}a_{13,1}a_{24,1}a_{24,2} + a_{13,1}a_{13,1}a_{24,2}a_{24,2} \\ = (a_{13,2}a_{24,1} - a_{13,1}a_{24,2})^2. \end{gathered}$$

The final simplification fits with the idea that this representation is essentially the product of two determinants. However, for what comes next we will want to remember, again, where these factors came from in the Young symmetrizer, so we prefer to write

$$a_{T_1} \circ b_{T_1}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = a_{1,2}a_{2,1}a_{3,2}a_{4,1} - a_{1,1}a_{2,2}a_{3,2}a_{4,1} - a_{1,2}a_{2,1}a_{3,1}a_{4,2} + a_{1,1}a_{2,2}a_{3,1}a_{4,2},$$

which is a slight abuse of notation, but still no loss of information since the first index of every variable in any given monomial is distinct, the indexing keeps track of the order, so symmetric, wedge, or tensor product all record the same thing.

A similar computation gives $c_{T_2}(a_1 \otimes a_2 \otimes a_3 \otimes a_4)$

$$a_{T_2} \circ b_{T_2}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = (a_{1,1}a_{3,2} - a_{1,2}a_{3,1})(a_{2,1}a_{4,2} - a_{2,2}a_{4,1})$$

$$= a_{1,2}a_{2,2}a_{3,1}a_{4,1} - a_{1,1}a_{2,2}a_{3,2}a_{4,1} - a_{1,2}a_{2,1}a_{3,1}a_{4,2} + a_{1,1}a_{2,1}a_{3,2}a_{4,2}.$$

Next one computes the same expressions for the other two factors and multiplies (tensor product, but disjoint symbols) to obtain $c_{T_1}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) \otimes c_{T_1}(b_1 \otimes b_2 \otimes b_3 \otimes b_4) \otimes c_{T_1}(c_1 \otimes c_2 \otimes c_3 \otimes c_4) =$

$$(a_{1,1}a_{2,2}-a_{1,2}a_{2,1})(a_{3,1}a_{4,2}-a_{3,2}a_{4,1})(b_{1,1}b_{2,2}-b_{1,2}b_{2,1})(b_{3,1}b_{4,2}-b_{3,2}b_{4,1})(c_{1,1}c_{2,2}-c_{1,2}c_{2,1})(c_{3,1}c_{4,2}-c_{3,2}c_{4,1})$$

which has $4^3 = 64$ terms in its expansion, that we will not write explicitly.

The next step is to un-weave the terms. To do this we take the each monomial $a_{1,i}b_{1,j}c_{1,k}$ and replace it with the tensor variable x_{ijk}, keeping the same coefficient. This operation can be seen as the operator $f \mapsto x_{i,j,k}(a_{1,i}b_{1,j}c_{1,k})^*(f)$, where * indicates contraction. This produces 64 terms like

$$a_{2,1}a_{3,2}a_{4,1}b_{2,1}b_{3,2}b_{4,1}c_{2,1}c_{3,2}c_{4,1}x_{2,2,2}$$

Then we do the same with each $a_{2,i}b_{2,j}c_{2,k}$, which happens to produce 0, indicating that the polynomial we are trying to find is just the zero polynomial. If instead we use $c_{T_1}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) \otimes c_{T_1}(b_1 \otimes b_2 \otimes b_3 \otimes b_4) \otimes c_{T_2}(c_1 \otimes c_2 \otimes c_3 \otimes c_4)$, and cary out the same procedure, at the first step we obtain 64 terms, then the next step produces 48 terms like

$$-2a_{3,2}a_{4,1}b_{3,2}b_{4,1}c_{3,1}c_{4,1}x_{1,2,2}x_{2,1,2}$$

The third step produces 40 terms like

$$-2a_{4,1}b_{4,1}c_{4,1}x_{1,2,2}x_{2,1,2}x_{2,2,1}$$

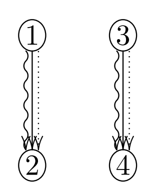
and the final step with $a_{4,i}b_{4,j}c_{4,k}$ one obtains (up to a factor of 12) the polynomial

$$\begin{gathered} x_{122}^2x_{211}^2 - 2x_{121}x_{122}x_{211}x_{212} + x_{121}^2x_{212}^2 - 2x_{112}x_{122}x_{211}x_{221} \\ -2x_{112}x_{121}x_{212}x_{221} + 4x_{111}x_{122}x_{212}x_{221} + x_{112}^2x_{221}^2 \\ +4x_{112}x_{121}x_{211}x_{222} - 2x_{111}x_{122}x_{211}x_{222} - 2x_{111}x_{121}x_{212}x_{222} \\ -2x_{111}x_{112}x_{221}x_{222} + x_{111}^2x_{222}^2. \end{gathered}$$

The shuffling laws can be understood via graphs, and there is a pictorial computation that one can do to see which of these triples of tableaux will produce a zero symmetrizer or not.

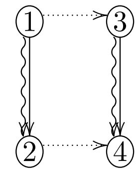
For the curious, the graphs are

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \quad \text{associated to}$$



(one condition to be non-zero from [Rai12] is that there should be no triangles when erasing arrow directions, which this graph fails, and hence is zero) and

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \text{associated to}$$



2.6. **Decomposition and computation for irreducible** $\mathrm{SL}(V)$ **modules inside** $\mathbb{C}[S^d(V)]$**.** Computing the $\mathrm{GL}(V)$ module description for $\mathbb{C}[S^d(V)]$ is similar to what we have done before, with a few notational changes. At first all we need to understand in order to know the representations is to compute the composition of Schur functors $S^e(S^dV)$, which can be done via characters for instance. But to do comptuations we would like to have a realization of the copies of these modules as modules of polynomials.

One approach to this is to use the concept of tabloids introduced by Raicu [Rai12]. We prefer to use fillings of tableau with indexed letters. We will say that a Young tableau T_π is d-standard if it is filled with the ordered alphabet $\{\mathrm{a}_1, \dots, \mathrm{a}_d, \mathrm{b}_1, \dots, \mathrm{b}_d, \cdots, \mathrm{c}_1, \dots, \mathrm{c}_d\}$ with no repetitions, and such that every column is strictly increasing and every row is strictly increasing. For example the following tableau is not d-standard $T = \begin{array}{|c|c|c|c|c|c|}\hline \mathrm{a}_1 & \mathrm{c}_1 & \mathrm{a}_2 & \mathrm{c}_2 & \mathrm{a}_3 & \mathrm{b}_3 \\ \hline \mathrm{b}_1 & \mathrm{d}_1 & \mathrm{b}_2 & \mathrm{d}_2 & \mathrm{c}_3 & \mathrm{d}_3 \\ \hline \end{array}$ but after permuting the entries in each row one obtains $T = \begin{array}{|c|c|c|c|c|c|}\hline \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 & \mathrm{b}_3 & \mathrm{c}_1 & \mathrm{c}_2 \\ \hline \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{c}_3 & \mathrm{d}_1 & \mathrm{d}_2 & \mathrm{d}_3 \\ \hline \end{array}$, which is d-standard.

Now we describe how to compute the Young symmetrizer associated with such tableaux using this example, which will produce an invariant for $S^4(S^3\mathbb{C}^2)$. As before, skew-symmetrize each column:

$$b_{(6,6)}(T) = (\mathrm{a}_1 \wedge \mathrm{b}_1) \otimes (\mathrm{a}_2 \wedge \mathrm{b}_2) \otimes (\mathrm{a}_3 \wedge \mathrm{c}_3) \otimes (\mathrm{b}_3 \wedge \mathrm{d}_1) \otimes (\mathrm{c}_1 \wedge \mathrm{d}_2) \otimes (\mathrm{c}_2 \wedge \mathrm{d}_3)$$

We compute the skew-symmetrizations via determinants:

$$(\mathrm{a}_1 \wedge \mathrm{b}_1) = \begin{vmatrix} \mathrm{a}_{1,1} & \mathrm{a}_{1,2} \\ \mathrm{b}_{1,1} & \mathrm{b}_{1,2} \end{vmatrix} = (\mathrm{a}_{1,1}\mathrm{b}_{1,2} - \mathrm{a}_{1,2}\mathrm{b}_{1,1})$$

So

$$b_{(6,6)}(T) = \begin{vmatrix} \mathrm{a}_{1,1} & \mathrm{a}_{1,2} \\ \mathrm{b}_{1,1} & \mathrm{b}_{1,2} \end{vmatrix} \begin{vmatrix} \mathrm{a}_{2,1} & \mathrm{a}_{2,2} \\ \mathrm{b}_{2,1} & \mathrm{b}_{2,2} \end{vmatrix} \begin{vmatrix} \mathrm{a}_{3,1} & \mathrm{a}_{3,2} \\ \mathrm{c}_{3,1} & \mathrm{c}_{3,2} \end{vmatrix} \begin{vmatrix} \mathrm{b}_{3,1} & \mathrm{b}_{3,2} \\ \mathrm{d}_{1,1} & \mathrm{d}_{1,2} \end{vmatrix} \begin{vmatrix} \mathrm{c}_{1,1} & \mathrm{c}_{1,2} \\ \mathrm{d}_{2,1} & \mathrm{d}_{2,2} \end{vmatrix} \begin{vmatrix} \mathrm{c}_{2,1} & \mathrm{c}_{2,2} \\ \mathrm{d}_{3,1} & \mathrm{d}_{3,2} \end{vmatrix},$$

which has $2^6 = 64$ terms when expanded. We consider the letters a_i to be commuting with each other, likewise for b_i, etc. We are interested in turning this expression into an expression using variables x_I with unordered indices I with $|I| = 3$ that represent the coefficients of a binary cubic. We do this in 4 steps, taking first every instance of $a_{1,i}a_{2,j}a_{3,k}$ and replacing it with $x_{\mathrm{sort}(i,j,k)}$. Doing this for each letter produces

$$3x_{1,1,2}^2x_{1,2,2}^2 - 4x_{1,1,1}x_{1,2,2}^3 - 4x_{1,1,2}^3x_{2,2,2} + 6x_{1,1,1}x_{1,1,2}x_{1,2,2}x_{2,2,2} - x_{1,1,1}^2x_{2,2,2}^2,$$

which is recognizable as the discriminant of the binary cubic, especially when one replaces x_I with $s_{\#|}$, where by $\#I$ we mean counting the number of 2's occurring:

$$3s_1^2s_2^2 - 4s_0s_2^3 - 4s_1^3s_3 + 6s_0s_1s_2s_3 - s_0^2s_3^2.$$

is the discriminant of the form $\sum_{k=0}^3 s_kx^k$.

2.7. **Decomposition and computation for irreducible $\mathrm{SL}(V)$ modules inside $\mathbb{C}[\bigwedge^d(V)]$.** The key to computing these modules, essentially we take the previous section and just transpose all the partitions, and many of the notations.

Again we will say that a Young tableau T_π is d-standard if it is filled with the ordered alphabet $\{\mathrm{a}_1, \dots, \mathrm{a}_d, \mathrm{b}_1, \dots, \mathrm{b}_d, \cdots, \mathrm{c}_1, \dots, \mathrm{c}_d\}$ with no repetitions, and such that every column is strictly increasing and every row is strictly increasing. For example the following tableau is not d-standard $T = \begin{array}{|c|c|}\hline \mathrm{a}_1 & \mathrm{b}_1 \\ \hline \mathrm{c}_1 & \mathrm{d}_1 \\ \hline \mathrm{a}_2 & \mathrm{b}_2 \\ \hline \mathrm{c}_2 & \mathrm{d}_2 \\ \hline \mathrm{a}_3 & \mathrm{c}_3 \\ \hline \mathrm{b}_3 & \mathrm{d}_3 \\ \hline \end{array}$ but after permuting the entries in each column one obtains $T = \begin{array}{|c|c|}\hline \mathrm{a}_1 & \mathrm{b}_1 \\ \hline \mathrm{a}_2 & \mathrm{b}_2 \\ \hline \mathrm{a}_3 & \mathrm{c}_3 \\ \hline \mathrm{b}_3 & \mathrm{d}_1 \\ \hline \mathrm{c}_1 & \mathrm{d}_2 \\ \hline \mathrm{c}_2 & \mathrm{d}_3 \\ \hline \end{array}$, which is d-standard.

Now we describe how to compute the Young symmetrizer associated with such tableaux using this example, which will produce an invariant for $S^4(\bigwedge^3\mathbb{C}^6)$. As before, skew-symmetrize each column by taking an associated determinant whose rows are given by the entries in the columns:

$$b_T = \begin{vmatrix} \mathrm{a}_{1,1} & \mathrm{a}_{1,2} & \mathrm{a}_{1,3} & \mathrm{a}_{1,4} & \mathrm{a}_{1,5} & \mathrm{a}_{1,6} \\ \mathrm{a}_{2,1} & \mathrm{a}_{2,2} & \mathrm{a}_{2,3} & \mathrm{a}_{2,4} & \mathrm{a}_{2,5} & \mathrm{a}_{2,6} \\ \mathrm{a}_{3,1} & \mathrm{a}_{3,2} & \mathrm{a}_{3,3} & \mathrm{a}_{3,4} & \mathrm{a}_{3,5} & \mathrm{a}_{3,6} \\ \mathrm{b}_{3,1} & \mathrm{b}_{3,2} & \mathrm{b}_{3,3} & \mathrm{b}_{3,4} & \mathrm{b}_{3,5} & \mathrm{b}_{3,6} \\ \mathrm{c}_{1,1} & \mathrm{c}_{1,2} & \mathrm{c}_{1,3} & \mathrm{c}_{1,4} & \mathrm{c}_{1,5} & \mathrm{c}_{1,6} \\ \mathrm{c}_{2,1} & \mathrm{c}_{2,2} & \mathrm{c}_{2,3} & \mathrm{c}_{2,4} & \mathrm{c}_{2,5} & \mathrm{c}_{2,6} \end{vmatrix} \cdot \begin{vmatrix} \mathrm{b}_{1,1} & \mathrm{b}_{1,2} & \mathrm{b}_{1,3} & \mathrm{b}_{1,4} & \mathrm{b}_{1,5} & \mathrm{b}_{1,6} \\ \mathrm{b}_{2,1} & \mathrm{b}_{2,2} & \mathrm{b}_{2,3} & \mathrm{b}_{2,4} & \mathrm{b}_{2,5} & \mathrm{b}_{2,6} \\ \mathrm{c}_{3,1} & \mathrm{c}_{3,2} & \mathrm{c}_{3,3} & \mathrm{c}_{3,4} & \mathrm{c}_{3,5} & \mathrm{c}_{3,6} \\ \mathrm{d}_{1,1} & \mathrm{d}_{1,2} & \mathrm{d}_{1,3} & \mathrm{d}_{1,4} & \mathrm{d}_{1,5} & \mathrm{d}_{1,6} \\ \mathrm{d}_{2,1} & \mathrm{d}_{2,2} & \mathrm{d}_{2,3} & \mathrm{d}_{2,4} & \mathrm{d}_{2,5} & \mathrm{d}_{2,6} \\ \mathrm{d}_{3,1} & \mathrm{d}_{3,2} & \mathrm{d}_{3,3} & \mathrm{d}_{3,4} & \mathrm{d}_{3,5} & \mathrm{d}_{3,6} \end{vmatrix}.$$

In this situation we view the variables a_i to be skew-commuting with each other, likewise for b_i, etc. This allows for one additional trick to be used, namely block Laplace expansion. In this example, let us use upper indices to indicate the columns we use for a determinant and the letters to indicate the row: For example

$$(\mathrm{a}_1 \wedge \mathrm{a}_2 \wedge \mathrm{a}_3)^{1,2,3} = \begin{vmatrix} \mathrm{a}_{1,1} & \mathrm{a}_{1,2} & \mathrm{a}_{1,3} \\ \mathrm{a}_{2,1} & \mathrm{a}_{2,2} & \mathrm{a}_{2,3} \\ \mathrm{a}_{3,1} & \mathrm{a}_{3,2} & \mathrm{a}_{3,3} \end{vmatrix} \quad \text{and} \quad (\mathrm{a}_1 \wedge \mathrm{a}_2 \wedge \mathrm{a}_3)^{4,5,6} = \begin{vmatrix} \mathrm{a}_{1,4} & \mathrm{a}_{1,5} & \mathrm{a}_{1,6} \\ \mathrm{a}_{2,4} & \mathrm{a}_{2,5} & \mathrm{a}_{2,6} \\ \mathrm{a}_{3,4} & \mathrm{a}_{3,5} & \mathrm{a}_{3,6} \end{vmatrix}.$$

So here is a hybrid expression for the first determinant in b_T has 18 terms:

$$(\mathrm{a}_1 \wedge \mathrm{a}_2 \wedge \mathrm{a}_3)^{1,2,3}(\mathrm{b}_3 \wedge \mathrm{c}_1 \wedge \mathrm{c}_2)^{4,5,6} \pm \cdots \pm (\mathrm{a}_1 \wedge \mathrm{a}_2 \wedge \mathrm{a}_3)^{4,5,6}(\mathrm{b}_3 \wedge \mathrm{c}_1 \wedge \mathrm{c}_2)^{1,2,3},$$

where the signs come from the Laplace transform as the indices run over 3 element subsets of $\{1, \dots, 6\}$. We can do another (block) Laplace expansion to handle the inner expressions:

$$\begin{aligned}(\mathrm{b}_3 \wedge \mathrm{c}_1 \wedge \mathrm{c}_2)^{1,2,3} &= (\mathrm{b}_3)^1(\mathrm{c}_1 \wedge \mathrm{c}_2)^{2,3} - (\mathrm{b}_3)^2(\mathrm{c}_1 \wedge \mathrm{c}_2)^{1,3} + (\mathrm{b}_3)^3(\mathrm{c}_1 \wedge \mathrm{c}_2)^{1,2}, \\ &= b_{3,1}\begin{vmatrix} \mathrm{c}_{1,2} & \mathrm{c}_{1,2} \\ \mathrm{c}_{2,3} & \mathrm{c}_{2,3} \end{vmatrix} - b_{3,2}\begin{vmatrix} \mathrm{c}_{1,1} & \mathrm{c}_{1,3} \\ \mathrm{c}_{2,1} & \mathrm{c}_{2,3} \end{vmatrix} + b_{3,3}\begin{vmatrix} \mathrm{c}_{1,1} & \mathrm{c}_{1,2} \\ \mathrm{c}_{2,1} & \mathrm{c}_{2,2} \end{vmatrix}.\end{aligned}$$

For another example

$$\begin{aligned}(\mathrm{b}_3 \wedge \mathrm{c}_1 \wedge \mathrm{c}_2)^{4,5,6} &= (\mathrm{b}_3)^4(\mathrm{c}_1 \wedge \mathrm{c}_2)^{5,6} - (\mathrm{b}_3)^5(\mathrm{c}_1 \wedge \mathrm{c}_2)^{4.6} + (\mathrm{b}_3)^6(\mathrm{c}_1 \wedge \mathrm{c}_2)^{4,5} \\ &= b_{3,4}\begin{vmatrix} \mathrm{c}_{1,5} & \mathrm{c}_{1,6} \\ \mathrm{c}_{2,5} & \mathrm{c}_{2,6} \end{vmatrix} - b_{3,5}\begin{vmatrix} \mathrm{c}_{1,4} & \mathrm{c}_{1,6} \\ \mathrm{c}_{2,4} & \mathrm{c}_{2,6} \end{vmatrix} + b_{3,6}\begin{vmatrix} \mathrm{c}_{1,4} & \mathrm{c}_{1,5} \\ \mathrm{c}_{2,4} & \mathrm{c}_{2,5} \end{vmatrix}.\end{aligned}$$

Similarly the second determinant in the expression of b_T can be written as:

$$(\mathrm{b}_1 \wedge \mathrm{b}_2 \wedge \mathrm{c}_3)^{1,2,3}(\mathrm{d}_1 \wedge \mathrm{d}_2 \wedge \mathrm{d}_3)^{4,5,6} \pm \cdots \pm (\mathrm{b}_1 \wedge \mathrm{b}_2 \wedge \mathrm{c}_3)^{4,5,6}(\mathrm{d}_1 \wedge \mathrm{d}_2 \wedge \mathrm{d}_3)^{1,2,3}.$$

We keep track of the symmetrizations in the same way as before - we do this with indices so we can keep track of where each variable and each term has come from.

An advantage of block Laplace expansion is that we are going to replace $(\mathrm{a}_1 \wedge\ \mathrm{a}_2 \wedge\ \mathrm{a}_3)^I$ with the tensor variable x_I which form a basis of $\bigwedge^3\mathbb{C}^6$, similarly for $(\mathrm{d}_1 \wedge\ \mathrm{d}_2 \wedge\ \mathrm{d}_3)^I$. It's a bit trickier to do this with the split forms like the b's and c's. However, after changing a^I's and d^I's to x_I's we get

$$\left(x_{1,2,3}(\mathrm{b}_3 \wedge \mathrm{c}_1 \wedge \mathrm{c}_2)^{4,5,6} \pm \cdots \pm x_{4,5,6}(\mathrm{b}_3 \wedge \mathrm{c}_1 \wedge \mathrm{c}_2)^{1,2,3}\right)$$
$$\cdot \left((\mathrm{b}_1 \wedge \mathrm{b}_2 \wedge \mathrm{c}_3)^{1,2,3} x_{4,5,6} \pm \cdots \pm (\mathrm{b}_1 \wedge \mathrm{b}_2 \wedge \mathrm{c}_3)^{4,5,6} x_{1,2,3}\right).$$

For the terms that are split between different coefficients, we replace, for instance, we take for $i < j < k$ each expression $(\mathrm{b}_3)^i(\mathrm{b}_1 \wedge \mathrm{b}_2)^{j,k} - (\mathrm{b}_3)^j(\mathrm{b}_1 \wedge \mathrm{b}_2)^{i,k} + (\mathrm{b}_3)^k(\mathrm{b}_1 \wedge \mathrm{b}_2)^{i,j}$ (which re-combines to $(\mathrm{b}_1 \wedge \mathrm{b}_2 \wedge \mathrm{b}_3)^{i,j,k}$) with $x_{i,j,k}$. We do the same with c's. Finally we produce the polynomial:

$$\begin{gathered}
-x_{1,2,3}^2x_{4,5,6}^2+2x_{1,2,3}x_{1,2,4}x_{3,5,6}x_{4,5,6}-2x_{1,2,3}x_{1,2,5}x_{3,4,6}x_{4,5,6}+2x_{1,2,3}x_{1,2,6}x_{3,4,5}x_{4,5,6}-2x_{1,2,3}x_{1,3,4}x_{2,5,6}x_{4,5,6}\\
+2x_{1,2,3}x_{1,3,5}x_{2,4,6}x_{4,5,6}-2x_{1,2,3}x_{1,3,6}x_{2,4,5}x_{4,5,6}+2x_{1,2,3}x_{1,4,5}x_{2,3,6}x_{4,5,6}-4x_{1,2,3}x_{1,4,5}x_{2,4,6}x_{3,5,6}+4x_{1,2,3}x_{1,4,5}x_{2,5,6}x_{3,4,6}\\
-2x_{1,2,3}x_{1,4,6}x_{2,3,5}x_{4,5,6}+4x_{1,2,3}x_{1,4,6}x_{2,4,5}x_{3,5,6}-4x_{1,2,3}x_{1,4,6}x_{2,5,6}x_{3,4,5}+2x_{1,2,3}x_{1,5,6}x_{2,3,4}x_{4,5,6}-4x_{1,2,3}x_{1,5,6}x_{2,4,5}x_{3,4,6}\\
+4x_{1,2,3}x_{1,5,6}x_{2,4,6}x_{3,4,5}-x_{1,2,4}^2x_{3,5,6}^2+2x_{1,2,4}x_{1,2,5}x_{3,4,6}x_{3,5,6}-2x_{1,2,4}x_{1,2,6}x_{3,4,5}x_{3,5,6}+2x_{1,2,4}x_{1,3,4}x_{2,5,6}x_{3,5,6}\\
-4x_{1,2,4}x_{1,3,5}x_{2,3,6}x_{4,5,6}+2x_{1,2,4}x_{1,3,5}x_{2,4,6}x_{3,5,6}-4x_{1,2,4}x_{1,3,5}x_{2,5,6}x_{3,4,6}+4x_{1,2,4}x_{1,3,6}x_{2,3,5}x_{4,5,6}-2x_{1,2,4}x_{1,3,6}x_{2,4,5}x_{3,5,6}\\
+4x_{1,2,4}x_{1,3,6}x_{2,5,6}x_{3,4,5}+2x_{1,2,4}x_{1,4,5}x_{2,3,6}x_{3,5,6}-2x_{1,2,4}x_{1,4,6}x_{2,3,5}x_{3,5,6}-2x_{1,2,4}x_{1,5,6}x_{2,3,4}x_{3,5,6}+4x_{1,2,4}x_{1,5,6}x_{2,3,5}x_{3,4,6}\\
-4x_{1,2,4}x_{1,5,6}x_{2,3,6}x_{3,4,5}-x_{1,2,5}^2x_{3,4,6}^2+2x_{1,2,5}x_{1,2,6}x_{3,4,5}x_{3,4,6}+4x_{1,2,5}x_{1,3,4}x_{2,3,6}x_{4,5,6}-4x_{1,2,5}x_{1,3,4}x_{2,4,6}x_{3,5,6}\\
+2x_{1,2,5}x_{1,3,4}x_{2,5,6}x_{3,4,6}+2x_{1,2,5}x_{1,3,5}x_{2,4,6}x_{3,4,6}-4x_{1,2,5}x_{1,3,6}x_{2,3,4}x_{4,5,6}+2x_{1,2,5}x_{1,3,6}x_{2,4,5}x_{3,4,6}-4x_{1,2,5}x_{1,3,6}x_{2,4,6}x_{3,4,5}\\
-2x_{1,2,5}x_{1,4,5}x_{2,3,6}x_{3,4,6}+4x_{1,2,5}x_{1,4,6}x_{2,3,4}x_{3,5,6}-2x_{1,2,5}x_{1,4,6}x_{2,3,5}x_{3,4,6}+4x_{1,2,5}x_{1,4,6}x_{2,3,6}x_{3,4,5}-2x_{1,2,5}x_{1,5,6}x_{2,3,4}x_{3,4,6}\\
-x_{1,2,6}^2x_{3,4,5}^2-4x_{1,2,6}x_{1,3,4}x_{2,3,5}x_{4,5,6}+4x_{1,2,6}x_{1,3,4}x_{2,4,5}x_{3,5,6}-2x_{1,2,6}x_{1,3,4}x_{2,5,6}x_{3,4,5}+4x_{1,2,6}x_{1,3,5}x_{2,3,4}x_{4,5,6}\\
-4x_{1,2,6}x_{1,3,5}x_{2,4,5}x_{3,4,6}+2x_{1,2,6}x_{1,3,5}x_{2,4,6}x_{3,4,5}+2x_{1,2,6}x_{1,3,6}x_{2,4,5}x_{3,4,5}-4x_{1,2,6}x_{1,4,5}x_{2,3,4}x_{3,5,6}+4x_{1,2,6}x_{1,4,5}x_{2,3,5}x_{3,4,6}\\
-2x_{1,2,6}x_{1,4,5}x_{2,3,6}x_{3,4,5}-2x_{1,2,6}x_{1,4,6}x_{2,3,5}x_{3,4,5}+2x_{1,2,6}x_{1,5,6}x_{2,3,4}x_{3,4,5}-x_{1,3,4}^2x_{2,5,6}^2+2x_{1,3,4}x_{1,3,5}x_{2,4,6}x_{2,5,6}\\
-2x_{1,3,4}x_{1,3,6}x_{2,4,5}x_{2,5,6}-2x_{1,3,4}x_{1,4,5}x_{2,3,6}x_{2,5,6}+2x_{1,3,4}x_{1,4,6}x_{2,3,5}x_{2,5,6}+2x_{1,3,4}x_{1,5,6}x_{2,3,4}x_{2,5,6}-4x_{1,3,4}x_{1,5,6}x_{2,3,5}x_{2,4,6}\\
+4x_{1,3,4}x_{1,5,6}x_{2,3,6}x_{2,4,5}-x_{1,3,5}^2x_{2,4,6}^2+2x_{1,3,5}x_{1,3,6}x_{2,4,5}x_{2,4,6}+2x_{1,3,5}x_{1,4,5}x_{2,3,6}x_{2,4,6}-4x_{1,3,5}x_{1,4,6}x_{2,3,4}x_{2,5,6}\\
+2x_{1,3,5}x_{1,4,6}x_{2,3,5}x_{2,4,6}-4x_{1,3,5}x_{1,4,6}x_{2,3,6}x_{2,4,5}+2x_{1,3,5}x_{1,5,6}x_{2,3,4}x_{2,4,6}-x_{1,3,6}^2x_{2,4,5}^2+4x_{1,3,6}x_{1,4,5}x_{2,3,4}x_{2,5,6}\\
-4x_{1,3,6}x_{1,4,5}x_{2,3,5}x_{2,4,6}+2x_{1,3,6}x_{1,4,5}x_{2,3,6}x_{2,4,5}+2x_{1,3,6}x_{1,4,6}x_{2,3,5}x_{2,4,5}-2x_{1,3,6}x_{1,5,6}x_{2,3,4}x_{2,4,5}-x_{1,4,5}^2x_{2,3,6}^2\\
+2x_{1,4,5}x_{1,4,6}x_{2,3,5}x_{2,3,6}-2x_{1,4,5}x_{1,5,6}x_{2,3,4}x_{2,3,6}-x_{1,4,6}^2x_{2,3,5}^2+2x_{1,4,6}x_{1,5,6}x_{2,3,4}x_{2,3,5}-x_{1,5,6}^2x_{2,3,4}^2
\end{gathered}.$$

3. DEGREE 6 BINARY INVARIANTS AND GRAPHS

There is a 4 dimensional space of $\mathrm{GL}(2)^{\times 4}$ invariants in degree 6. In this note we compute a basis of this space.

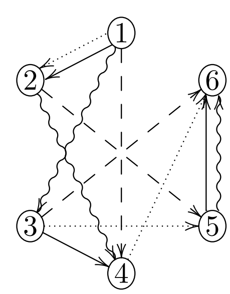
The tableaux associated to the partition $(3, 3)$ has 5 standard fillings:

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}$$

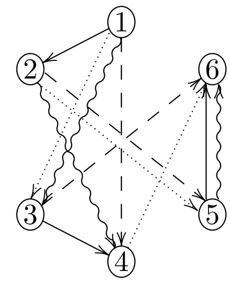
A basis of the highest weight space for $S_{3,3}\mathbb{C}^2 \otimes S_{3,3}\mathbb{C}^2 \otimes S_{3,3}\mathbb{C}^2 \otimes S_{3,3}\mathbb{C}^2$ is constructed by considering the images of the Young Symmetrizers prescribed by 4 tableau. One way to select a good 4-tuple of tableau so that the images are linearly independent is by fixing the first tableau, and choosing 3 of the remaining 4 (in 4 different ways). Moreover, to each 4-tuple of tableau we may associate a colored graph (see Raicu GSS), where color 1 corresponds to $\longrightarrow$, color 2 to $\rightsquigarrow$, color 3 to $\dashrightarrow$ and color 4 to $\cdots\!\!\!\rightarrow$.

A choice of 4 basis elements is described by

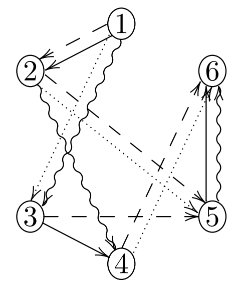
$$T = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}$$ is associated to



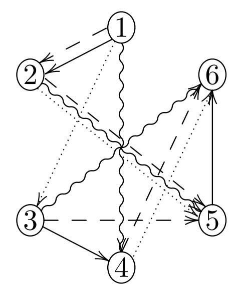
$$T = \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4&6\\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4&6\\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5&6\\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5&6\\ \hline\end{array} \quad \text{is associated to}$$



$$T = \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4&6\\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4&6\\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5&6\\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5&6\\ \hline\end{array} \quad \text{is associated to}$$



$$T = \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4&6\\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5&6\\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5&6\\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5&6\\ \hline\end{array} \quad \text{is associated to}$$



We can refine this discussion even more by realizing that the multiplicity space $[(3,3)]^{\otimes 4}$ is a representation of $\S_4$ also. So inside the 5^4-dimensional space spanned by the tensor products of standard tableaux of shape $(3,3)$ there is this special 4-dimensional subspace that is spanned by taking tensor products of distinct tableaux. We would like to understand this better.

4. Other ways to compute equations

The methods we described up to now allow us to do interpolation and find all equations of low degree vanishing on an algebraic variety with the kinds of symmetry we're interested in. There are other methods that present as *ad hoc*, but nonetheless can be useful because they either have a geometric interpretation that we can exploit, or they are easy to compute, or both.

One such method is that of Young flattenings, which was invented by Landsberg and Ottaviani in [LO11].

4.0.1. *Kostant's Theorem for $\mathcal{I}(G/P)$ in the Segre, Veronese, and Grassmann cases, and conjecture / question about $\mathcal{I}(\sigma_2(G/P))$.*

References

[BO11] D. J. Bates and L. Oeding, *Toward a salmon conjecture*, Exp. Math. **20** (2011), no. 3, 358–370. `arXiv:1009.6181`. ↑2.5.2

[FH91] W. Fulton and J. Harris, *Representation theory, a first course (graduate texts in mathematics)*, Graduate Texts in Mathematics, vol. 129, New York: Springer-Verlag, 1991. ↑1

[GW98] R. Goodman and N. Wallach, *Representations and invariants of the classical groups*, Encyclopedia of Mathematics and its Applications, vol. 68, Cambridge University Press, 1998. ↑1, 2.18

[HIL13] Jonathan D Hauenstein, Christian Ikenmeyer, and Joseph M Landsberg, *Equations for lower bounds on border rank*, Experimental Mathematics **22** (2013), no. 4, 372–383. ↑2.1.1

[Lan12] J.M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence, RI, 2012. ↑1, 2.22

[LO11] J.M. Landsberg and G. Ottaviani, *Equations for secant varieties of Veronese and other varieties*, Ann. Mat. Pura Appl. (4) (2011), 1–38 (English). `arXiv:1111.4567`. ↑4

[OR14] L. Oeding and C. Raicu, *Tangential varieties of Segre-Veronese varieties*, Collect. Math. (2014), 1–28. `arXiv:1111.6202`. ↑2.5.2

[OS16] L. Oeding and S. V Sam, *Equations for the fifth secant variety of Segre products of projective spaces*, Experimental Mathematics **25** (2016), 94–99. `arXiv:1502.00203`. ↑2.1.1, 2.5.2

[Rai12] C. Raicu, *Secant varieties of Segre-Veronese varieties*, Algebra Number Theory **6-8** (2012), 1817–1868. `arXiv:1011.5867`. ↑2.5.2, 2.6