Categorical Quotient (affine case)

G be reductive (affine) algebraic group

e.g. $\text{GL}_n(\mathbb{C})$, $\text{SL}_n(\mathbb{C})$

X affine variety on which G acts morphically

Then the invariant ring $\mathbb{C}[X]^G$ is finitely generated. (Hilbert)

We have an inclusion of affine algebras:

The corresponding map of varieties is constant on orbits.

What is the description of the variety corresponding to the invariant ring?

What is the morphism of varieties corresponding to the above inclusion of algebras?

$\text{Max Spec } (\mathbb{C}[X]^G) \subset X$
Affine alg group. Group object in the category of affine varieties.

Over \( \mathbb{C} \): reductive \( \Rightarrow \) linearly reductive

Every \((f,d)\) representation is a direct sum of irreducible representation.

\( \Rightarrow \) Every subrepresentation has a complement. \[ W \subseteq V \text{ is invariant then } \exists W' G\text{-invariant s.t. } W \oplus W' = V. \]

\[ G \mathfrak{V} X; \quad G \mathfrak{U} k[ X ] \]

\( \{ f \in k[ X ] \} = k[ X ]^G \subseteq k[ X ] \)

\( g = g' \) \( \iff \) \( f(x) = f(G^{-1} x) \) affine algebras

affine varieties and reduced affine algebra
$X \rightarrow R[x] := \mathbb{C}[x_1, \ldots, x_n]$ 

Max ideals of $R[x]$ 

Max Spec $(R[x]) = \text{set of closed } G\text{-orbits in } X$

$G \rightarrow X$ that closed orbit $\Rightarrow \overline{Gx}$ contains a unique closed orbit
Example:

$\text{Max Spec}(k) = \{1, 3\}

\mathbb{C}^n = \mathbb{C}^n(n \in \mathbb{N})$

$K = k[x] \subseteq k[x][y, z]$

$G \lhd \mathbb{C}$

$G \mathbb{C}$

$G \mathbb{C} = \{1, 0, \ldots, 0\}$

$\text{G-orbits of } x$

$\mathbb{C}^n$

$G \mathbb{C}$

$\text{single point}$
$\mathbb{C}^* \text{ acting on } \mathbb{P}^1 = \mathbb{C} \cup \{ \infty \}$

$\mathbb{C}^* \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$

$t \cdot ([x:y]) = [tx:y] = [x:ty]\]

$\{ [x:y] \mid x \neq 0, y \neq 0 \}$ is an orbit

$0 \cdot [0:1]$
$\infty \cdot [1:0]$

$\mathbb{C}^* \text{ acting on } \mathbb{C}^2 = X$

$t \cdot (x,y) = (tx, ty)$

$\mathbb{C}[y] \hookrightarrow \mathbb{C}[x, y]$

$\mathbb{C}[xy] \subseteq \mathbb{C}[x, y]$
$GL_n(\mathbb{C})$ acting by conjugation on $M_n(\mathbb{C})$

Set of eigenvalues of $x$: $\mathbb{K}[x]$ of $G$

$\mathbb{K}[\text{Coeffs of the char. polynomial}]$

$\mathbb{K}[\text{Symmetric funs in the eigenvalues}]$

$\mathbb{C}[\text{elem symm} \leq \mathbb{C}[\lambda_1, \ldots, \lambda_n]^G]$

$\mathbb{C}[\text{elem symm} \leq \mathbb{C}[\lambda_1, \ldots, \lambda_n]^{\text{eigenvalues}}$
Proposition: \( G \) reductive, \( X \) affine \( G \)-variety

Let \( Y \) and \( Z \) be closed \( G \)-invariant disjoint subsets of \( X \).

Then there exists function \( f \) on \( X \), invariant under \( G \), with \( f(Y) = 0 \) and \( f(Z) = 1 \).

**Proof:**

\( J = \text{ideal of } Y \)
\( K = \text{ideal of } Z \)

\( J + K = k[x] \)

\( \emptyset + K = 1 \)

\( f(z) = 1 \forall z \in Z \)

\( \mathbb{R}^G \) does the job.

Propn: \( G \) reductive, \( X \) affine \( G \)-variety

Every \( G \)-orbit closure has a unique closed orbit.
Some representation theory:

G-linear map

\[ V, W \]
\[ V \rightarrow V \text{ is linear} \quad \forall g \in G \]

\[ V \xrightarrow{\varphi} W \quad G \text{-map} \]

\[ \varphi(gx) = g(\varphi x) \]

Schur’s Lemma:

\[ V \xrightarrow{\varphi} W \text{ irreducible } G \text{-repns}, \quad \varphi \text{ is } G \text{-map} \]

Then \( \varphi \) is an isomorphism or \( \varphi = 0 \)

\[ V \text{ f.d. irreducible } G \text{-rep.} \text{ Then } \text{End}_G(V) = \mathbb{C} \]

\[ \exists \varphi: V \rightarrow V \mid \varphi \text{ is linear} \]

\[ \varphi \text{ is a } G \text{-map} \]

The multiplicity of any given simple (=irreducible) module in a semisimple module is uniquely defined (independent of the decomposition).

\[ V = \text{direct sum of irreps} \]

Then this decomposition is unique.

\[ V = \mathbb{S}_0 \oplus \mathbb{S}_2 \oplus \mathbb{S}_0 \oplus \mathbb{S}_2 \]

\[ \mathbb{S}_2 \oplus \mathbb{S}_2 \]

\[ \mathbb{S}_0 \oplus \mathbb{S}_2 \]

\[ \mathbb{S}_2 \oplus \mathbb{S}_0 \]
Matsushima’s theorem (a variant)  
G reductive,  
X affine G-variety on which the action of G is transitive  
Then the stabiliser of any point of X is reductive 

\[ G_x := \{ g \in G \mid gx = x \} \text{ closed subgp of } G \]
Example:

\( \text{GL}_2(\mathbb{C}) \) acting on \( \text{M}_2(\mathbb{C}) \) by conjugation.

Nilpotent matrices \( \mathfrak{so}_3 \) is an orbit.

It is not an affine space.

Closed orbits are those containing diagonal matrices.

\[ \begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]

\( \text{Stab}(\mathfrak{n}) \), \( \text{GL}_2(\mathbb{C}) \) \& Nilpotent matrices

\( 2 \text{R} \times \mathbb{C} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \)

\( \{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{C}, a \neq 0 \} \)
Homogeneous spaces and their structure:

(At the level of varieties.)

Proposition: G affine algebraic group, H closed subgroup. Then there exists a finite dimensional (rational) representation $V$ of $G$ and a vector $v$ in $V$ such that $H = \{ g \in G \mid gv \in \langle v \rangle \}$
At the level of sets: \[ G \times X \text{ transitive} \]

\[ G \overset{G_x}{\underset{G}{\times}} G_x \quad \forall x \in X \]

\[ G \overset{G_x}{\underset{G}{\times}} G \text{-set} \]

\[ a_1 \leq n_2 \leq n \]

\[ U \subseteq W \subseteq C^n \]

Examples: Projective space, Grassmannians, flag varieties

\[ GL_n \overset{U \subseteq C^n}{\underset{\text{sub}}{\times}} \text{2-dim space in } C^n \]

\[ GL_n \overset{\text{stabs}}{\underset{\text{of some point}}{\times}} \text{Grassmannian} \]

\[ GL_n = \text{stabs of some point} \]
More Examples from Classical Invariant Theory

\[ \mathfrak{sp}(V) \cong V \oplus m \]

\[ \mathfrak{gl}(V) \cong V \otimes V^* \oplus \mathfrak{g} \]

\[ \mathfrak{g} \cong \nu_1 \cdots \nu_m \times \mathfrak{g}_1 \cdots \mathfrak{g}_2 \]

\[ k[x] \mathfrak{g} \subset k[x] \]

\[ \mathfrak{s}[\nu_i, \theta_j] \mid 1 \leq i \leq m, 1 \leq j \leq 2 \]

\[ \mathfrak{g} \cong \mathfrak{sp}(V) \cong V \oplus m \]

\[ \mathfrak{so}(V) \cong V \oplus m \]

\[ \text{space of all } m \times 2 \text{ matrices of rank } \leq n \]
$\text{SL}(n) \cong V^\oplus m \oplus V^* \oplus \mathbb{O}$

$\mathbb{K}[X]^G \cong \mathbb{C} \left[ \langle v_i, f_j \rangle \left| \begin{array}{l} 1 \leq i \leq m \\ 1 \leq j \leq q \end{array} \right. \right]$

$\eta = \dim V$

$\det = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \left| \begin{array}{cc} 1 \leq i < j \leq m \end{array} \right.$

$\det = \begin{bmatrix} f_1 \cdots f_q \end{bmatrix} \left| \begin{array}{cc} 1 \leq i < j \leq q \end{array} \right.$