https://www.imsc.res.in/~knr/past/lag13/index.html

Videos of lectures of a first course on linear algebraic groups (from Springer)

Categorical Quotient (affine case)

G be reductive (affine) algebraic group

e.g. $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$

$G \times X \to X$

X affine variety on which G acts morphically

Then the invaring ring $k[X]^G$ is finitely generated as a $\mathbb{C}$-alg. (Hilbert)

We have an inclusion of affine algebras:

The corresponding map of varieties is constant on orbits.

What is the description of the variety corresponding to the invariant ring?

What is the morphism of varieties corresponding to the above inclusion of algebras?

$$\mathrm{Max\,Spec}(k[X]^G) \xleftarrow{?} X$$

Affine alg group: Group object in the category of affine varieties.

Over $\mathbb{C}$: reductive $\Rightarrow$ linearly reductive

Every (f.d) representation is a direct sum of irreducible representation.

$\Rightarrow$ Every subrepresentation has a complement. [$W \subseteq V$ is invariant, then $\exists\, W'$ G-invariant s/t $W \oplus W' = V$.]

$G \circlearrowright X$; $G \circlearrowright k[X]$ $\quad ({}^g f)(x) = f(g^{-1}x)$

$\{f \in k[X] \mid {}^g f = f\} =: k[X]^G \subseteq k[X]$ affine algebras

1-1 correspondence between affine varieties and reduced affine algebras

$$X \longmapsto k[X] := \frac{\mathbb{C}[x_1, \dots, x_n]}{I(X)}$$

$X \subseteq \mathbb{C}^n$

$$\text{Max ideals of } k[X] \longleftarrow\!\shortmid\; k[X]$$

$$f(gx) = \underbrace{(g^{-1}f)}_{=f}(x) = f(x)$$

$f(gx) = f(x')$

$\mathrm{Max\,Spec}(k[X]^G) =$ set of closed G-orbits in X

$$G /\!\!/ X \longleftarrow X$$

that closed orbit $\longleftarrow\!\shortmid$ $\overline{Gx}$ contains a unique closed orbit

Example: $GL_n(\mathbb{C})$ acting on $\mathbb{C}^n$

$$\mathbb{C}^n = X$$

$\text{Max-Spec}(k) = \{*\}$

$$k = k[X]^{GL_n(\mathbb{C})} \subseteq k[X] \qquad (1, 0, \dots, 0)$$

$$G \backslash\backslash X = \text{single point}$$

$$G \backslash X = G\text{-orbits of } X$$

$$\begin{tikzcd} & G\backslash X \arrow[dl] & \\ G\backslash\backslash X & & X \arrow[ul] \arrow[ll] \end{tikzcd}$$

$\mathbb{C}^\times$ acting on $\mathbb{P}^1 = \mathbb{C}\cup\{\infty\}$ $\quad [x:y]$

$\mathbb{C}^\times \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$

$t\cdot([x:y]) = [tx:y] = [x:t^{-1}y]$

$\{[x:y] \mid x\neq 0,\ y\neq 0\}$ is an orbit

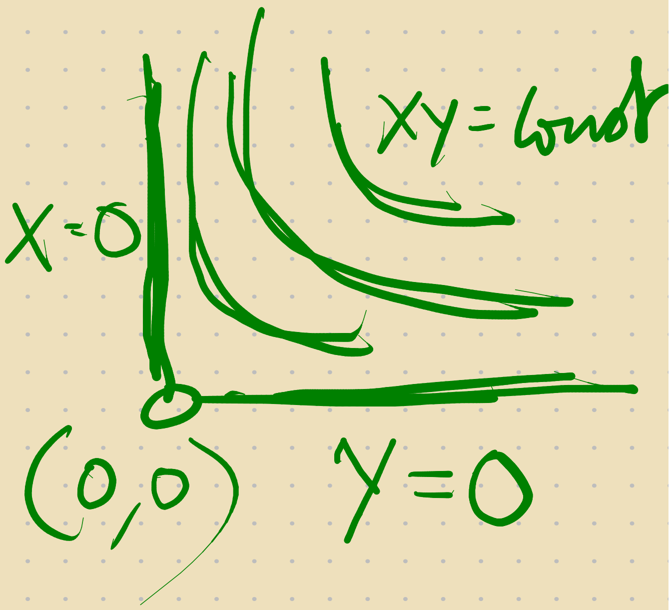
$\mathbb{C}^\times$

$0 \quad [0:1]$

$\infty - [1:0]$

$\mathbb{C}^\times$ acting on $\mathbb{C}^2 = X$

$t\cdot(x,y) = (tx, t^{-1}y)$



$\mathbb{C}[u] \qquad k[X]$

$\mathbb{C}[xy] \subseteq \overset{\parallel}{\mathbb{C}[x,y]}$

$GL_n(\mathbb{C})$ acting by conjugation on $M_n(\mathbb{C})$

$$GL_n(\mathbb{C}) = G, \qquad M_n(\mathbb{C}) = X$$

$$G\backslash\backslash X \longleftarrow X$$

$G\backslash\backslash X$ = set of eigenvalues of X

$$G\backslash\backslash X = k[X]^G \subseteq k[M_n(\mathbb{C})] = k\begin{bmatrix} x_{11}, \dots, x_{1n} \\ x_{21}, \dots, x_{2n} \\ \vdots \\ x_{n1}, \dots, x_{nn} \end{bmatrix}$$

$$k[X]^G = k[\text{Coeffs of the char. polynomial}]$$

$$= \mathbb{C}[\text{Symmetric fns in the eigenvalues}] = \mathbb{C}[\lambda_1, \dots, \lambda_n]^{\mathfrak{S}_n}$$

$$\mathbb{C}[\text{elem symm fns}] \subseteq \mathbb{C}[\underbrace{\lambda_1, \dots, \lambda_n}_{\text{eigenvalues}}]$$

$\mathbb{C}[\text{elem symm fns}]$ = polynomial ring

Proposition: G reductive, X affine G-variety

Let Y and Z be closed G-invariant disjoint subsets of X.

$Y \cap Z = \text{empty}$

Then there exists function f on X, $\boxed{\text{invariant under } G}$, with f(Y)=0 and f(Z)=1.

Proof:

$J = \text{ideal of } Y$

$K = \text{ideal of } Z$

$J + K = k[X]$

$f + k = 1$

$f(z) = 1 \;\; \forall z \in Z$

$\boxed{Rf}$ does the job.

$J + K \subseteq k[X]$

$\cap\!\!|$

$m_x \quad \underbrace{x \in X}$

$x \in Y \cap Z$

$\Rightarrow \Leftarrow$

$$\begin{array}{ccc} k[X]^G & \xleftarrow{R_X} & k[X] \\ \downarrow & & \downarrow \\ k[Y]^G & \xleftarrow{R_Y} & k[Y] \end{array}$$

Propn: G reductive, X affine G-variety

Every G-orbit closure has a unique closed orbit.

$\subseteq \overline{Gx}$:

Some representation theory:

$v \mapsto gv$

$V \to V$ is linear $\forall g \in G$

G-linear map V, W

$V \xrightarrow{\varphi} W$ G-map

$\varphi(gx) = g(\varphi x)$

Schur's Lemma:

$V \xrightarrow[\varphi]{} W$ irred f.d. G-repns; φ G-map linear

Then φ is an isomorphism or $\varphi = 0$

V f.d. irred G-rep. Then $\underbrace{\mathrm{End}_G(V)}_{\{\varphi : V \to V \mid \varphi \text{ is linear}, \ \varphi \text{ is a G-map}\}} = \mathbb{C}$

$\varphi : V \to V$.

The multiplicity of any given simple (=irreducible) module in a semisimple module is uniquely defined (independent of the decomposition).

V = direct sum of irreps

Then this decomposition is unique

$\mathrm{End}_G(V) = \underbrace{\mathrm{End}(V)^G}$

$V = S_1 \oplus S_1 \oplus S_2 \oplus S_2 \oplus S_2$

$\quad = S_1 \oplus S_2 \oplus S_1 \oplus S_2 \oplus S_2$

Matsushima's theorem (a variant) G reductive,

X affine G-variety on which the action of G is transitive

Then the stabiliser of any point of X is reductive

$G_x := \{g \in G \mid gx = x\}$ closed subgp of G

Example: $GL_2(\mathbb{C})$ acting on $M_2(\mathbb{C})$ by conjugation.

Nilpotent matrices $\setminus \{0\}$ is an orbit

It is not an affine space.

Closed orbits are those containing diagonal matrices

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$GL_2(\mathbb{C}) \curvearrowright$ Nilpotent matrices

2 orbits $\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{= N}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\mathrm{Stab}(N) = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \,\middle|\, \begin{array}{l} a, b \in \mathbb{C} \\ a \neq 0 \end{array} \right\}$$

Homogeneous spaces and their structure:

(At the level of varieties.)

$$G/G_x \simeq Gx$$

G affine alg gp, H closed subgp G/H is a variety.

Proposition: G affine algebraic group, H closed subgroup
Then there exists a finite dimensional (rational) representation V of G and a vector v in V such that H={g in G | gv \in <v>}

$$gv \in \mathbb{C}v$$

At the level of sets: $G \circlearrowright X$ transitive

$x \in X$

$G \circlearrowright G/G_x \xrightarrow{\sim} Gx$

G-set

$n_1 \leq n_2 \leq n$

$V \subseteq W \subseteq \mathbb{C}^n$

Examples: Projective space, Grassmannians, flag varieties

$GL_n \circlearrowright \mathbb{C}^n$

$GL_n \circlearrowright \mathbb{P}^{n-1}$

transitive

$$\frac{GL_n}{\text{Stab of some point}} = \mathbb{P}^{n-1}$$

$GL_n \circlearrowright \mathbb{C}^n$

2-dim subspaces in $\mathbb{C}^n$

$$\frac{GL_n}{\text{Stab.}} \simeq \text{Grassmannian}$$

More Examples from Classical Invariant Theory

$$\underbrace{GL_n(V)}_{\shortparallel\ G} \circlearrowleft \underbrace{V^{\oplus m} \oplus V^{*\oplus q}}_{X}$$

$$v_1, \dots, v_m \quad f_1, \dots, f_q$$

$$k[X]^G \subseteq k[X] \qquad \underline{n = \dim V}$$

$$\underline{\mathbb{C}[\langle v_i, f_j \rangle} \mid 1 \le i \le m,\ 1 \le j \le q]$$

$$Sp(V) \circlearrowleft V^{\oplus m}$$

$$O(V) \circlearrowleft V^{\oplus m}$$

$$X /\!\!/ G$$

space of all $m \times q$ matrices of rank $\le n$

$$\underbrace{SL(V)}_{G} \circlearrowright \underbrace{V^{\oplus m} \oplus V^{*\oplus q}}_{X}$$

$$k[X]^G = \mathbb{C}\left[\langle v_i, f_j \rangle \;\middle|\; \begin{array}{l} 1 \le i \le m \\ 1 \le j \le q \end{array} \right.$$

$n = \dim V$

$$\overset{\det}{=} [v_{i_1}, \dots, v_{i_n}] \mid 1 \le i_1 < \dots < i_n \le m$$

$$\overset{\det}{=} [f_{j_1}, \dots, f_{j_n}] \mid 1 \le j_1 < \dots < j_n \le q \Big]$$