Representations, Reductive Groups, Invariants

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Orientation

- The overall goal is to see how geometry can serve needs of complexity theory. (Item 4 of first lecture, serving first 3 items)
- Today: how representation theory can be useful to study the relevant geometry. (Item 5 of first lecture)
- If you are new to this, please focus on definitions and EXAMPLES. (Work out small examples for the symmetric and general linear groups.)
- It is all very concrete, "just" linear and multilinear algebra.

Overview

- Action of a group on a set
- Linear action of a group on a vector space = "Group representation"
- Reductive algebraic groups, complete reducibility
- Reductive group acting on an affine variety, linearization
- Idea of a quotient, ring of invariants, finite generation

Group action on a set

Group G acting on a set S means

Action map	currying	Group homomorphism
$G \times S \to S$	$\rightsquigarrow$	$G \to$ permutations of S

Both ways of thinking are useful. Action map is used more often, with the "dot" notation:

$$(g, x) \mapsto g \cdot x \qquad \text{By definition it satisfies} \qquad g \cdot (h \cdot x) = (gh) \cdot x \qquad \text{and} \qquad 1 \cdot x = x$$

This gives the equivalence relation $x \sim g \cdot x$, leading to partition of S into equivalence classes

$$S = \coprod \text{orbits} \qquad \text{where orbit of } x \in S \text{ is } G \cdot x$$

Examples of group action on a set

Recall definition: $G \times S \to S$ such that $g\cdot(h\cdot x) = (gh)\cdot x$ and $1\cdot x = x$.

- Trivial action of any G on any S. This means $g\cdot x = x$ for all $g \in G$ and $x \in S$.
- Defining action of the symmetric group: S_n acts naturally on $\{v_1, \ldots, v_n\}$.

New actions from old: suppose G acts on S. Then G also acts on

- $S \times S$, $S \times S \times S, \ldots,$ i.e. on lists of elements from S (of a fixed length)
- subsets of S (of fixed cardinality)
- multisets made of elements of S (of fixed cardinality)

Exercise: apply this to the natural S_n action and work out orbits for small length/cardinality. Formulate graph isomorphism question in this language.

Group representations

"G acts linearly on vector space V or V is a representation of G or V is a G-module" means

Linear action map	curying	Group homomorphism
$G \times V \to V$	$\rightsquigarrow$	$G \to GL(V)$

Blanket assumption for us: V will be a $\mathbb{C}$-vector space and (almost always) finite dimensional.

For now take on faith that group representations are useful! How to analyze them?

- We again have orbit decomposition of V. Often this is of great interest.
- But we are also interested in a linear decomposition of V into smaller representations.

  Subrepresentation or G-submodule of V = subspace of V that is stable under action of G.

Let's first look at some examples of representations ...

Examples of representations

- Trivial action: $g \cdot v = v$ for all $g \in G$ and $v \in V$ (any group G and any vector space V)
- Can linearize action of G on any set S. For example

  S_n action on set $\{v_1, \ldots, v_n\}$ $\rightsquigarrow$ *Linear* S_n action on $V = \{\sum_{i=1}^{n} c_i v_i \mid c_i \in \mathbb{C}\}$.

  Can think of linearization = functions on the set S. This idea will be useful again later!
- New representations from old
  - Direct sum: linear action of G on V and $W \rightsquigarrow$ action on $V \oplus W$ by block diagonal matrices.
  - Subrepresentations of a representation: a subspace stable under action of G

Exercise: The S_n-module $V = \{\sum c_i v_i\}$ has exactly two (proper nonzero) subrepresentations and V is their direct sum. General result of this nature?

Decomposing representations into irreducibles

A representation V is called irreducible or simple if it has no proper nonzero subrepresentation.

Desirable situation for a group G

1. We can classify its irreps (even better if we can construct them and compute explicitly)
2. Every representation is a direct sum of irreps (called completely reducible or semisimple)

- Semisimplicity is equivalent to saying that every subrepresentation has a complement.
- Theorem: desirable situation is always true for a finite group G. For $G = S_n$, we have an explicit story. Irreps are in bijection with partitions of n and one can construct them.
- The simple submodules in the decomposition are not unique but the isotypic components ($=$ sum of all isomorphic irreducible subreps) are (like eigenspaces of a matrix)

Need more examples to see this in action ...

More representations using multilinear algebra

If G acts linearly on V, then G also acts linearly on a vector space whose basis is

- ordered pairs (or triples, ...) of a basis of V, i.e., tensor powers $V \otimes V$, $V \otimes V \otimes V, \ldots$
- degree d monomials using basis vectors of V as variables, i.e., symmetric powers $Sym^d V$
- dual to that of V, i.e. dual space V^*. Check that $g \cdot f(v) := f(g^{-1} \cdot v)$ is an action.

Exercise: decompose second tensor/symmetric powers of the defining representation $\mathbb{C}^n$ of S_n.

Representations built out of this machinery are very relevant for symbolic computation (so I am told). They will feature prominently when we go the representations of the general linear group. Now back to algebraic groups ...

Back to varieties and algebraic groups

Affine variety X = common zeros of a set of polynomials in $\mathbb{C}[t_1, \dots t_n]$

- This set can be taken to be a radical ideal I.
- Coordinate ring of X or ring of regular functions on X is $\mathbb{C}[X] = \mathbb{C}[t_1, \dots t_n]/I$.
- The geometry of X is completely determined by the ring $C[X]$.
- Morphism $X \to Y$ of affine varieties is determined by the ring map $\mathbb{C}[Y] \to \mathbb{C}[X]$.

Affine algebraic group G is a group and an affine variety in a compatible way.

- Multiplication $G \times G \to G$ and inverse $G \to G$ maps are morphisms of varieties.
- We want to consider only rational representations V, which means: the action map $G \times V \to V$ is also a morphism of varieties.

A lot of words to digest the first time around, but I was told a secret ...

Groups of interest

For us algebraic groups of interest are mostly $GL_n(\mathbb{C})$ and its relatives: closed subgroups like

- $SL_n(\mathbb{C})$
- $D_n =$ the diagonal matrices $\mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (n times)
- $B_n =$ Upper triangular matrices , $U_n =$ only those with 1's on the diagonal, and
- Products of such groups

Exercise: show these are affine algebraic groups and find their coordinate rings.

Note: $GL_n(\mathbb{C})$ is already a group of matrices, with its defining representation on $\mathbb{C}^n$. We still very much want to study its rational representations in other linear spaces i.e. group homomorphisms $GL_n(\mathbb{C}) \to GL_m(\mathbb{C})$ where entries in the latter matrix are polynomials in entries of the source matrix (possibly multiplied by negative powers of the determinant).

Reductive algebraic groups

There is a structural definition of reductive group, but we are happy to accept these theorems:

- Rational representations of reductive groups are completely reducible.
- $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$ and their products are reductive.
- We know their irreps explicitly. They are classified by their highest weight.

Exercise: show that D_n is reductive and find its irreps. (You may use that D_n acts diagonally on every rational representation.) Show that U_n is not reductive. (For a matrix with a repeated eigenvalue, the vector space need not have an eigenbasis.)

Longer exercise: Show that $Sym^2(V)$ is an irrep for $GL(V)$. What about $Sym^3(V)$? Decompose the $SL(V)$-module $V \otimes V$ into a direct sum of irreps. Can you do the same for $Sym^2(Sym^2(\mathbb{C}^2))$? This is the plethysm representation from lecture 1. (Strictly speaking it was $Sym^2(Sym^2(\mathbb{C}^2)^*)$: with x, y a basis of $\mathbb{C}^2$, so that x^2, xy, y^2 is a basis of $Sym^2(\mathbb{C}^2)$, these latter basis vectors were identified with their coefficients in a general linear combination.)

Back to orbits

- Now suppose a reductive G acts on an affine variety X. We want to study G-orbits and their closures.

  Note that this could even be a linear action on a vector space X (e.g. $\mathbb{C}^*$ acting on $\mathbb{C}$ by multiplication). But now we want to focus on the geometry of orbits (in the example the representation $\mathbb{C}$ is irreducible, but there are two orbits.)

- Orbit closures are affine varieties too. How do representations help? One can linearize the situation by considering <u>regular</u> functions on X:

$$G \text{ acts linearly on } \mathbb{C}[X] \text{ by } g \cdot f(x) = f(g^{-1} \cdot x)$$

- Easy check: For any regular function f on X, the linear span of $\{g \cdot f \mid g \in G\}$ is a finite dimensional rational representation of G. Use this to linearize the G-action on X: by taking finitely many of these G-stable subspaces generating $\mathbb{C}[X]$ as a ring, embed X as a closed subset of the span of these subspaces (embedding is compatible with G-actions).

Invariant ring of a group action

For a G-module V, the invariants are $V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}$.

For G acting on an affine variety X, the invariants $\mathbb{C}[X]^G$ form a subring of $\mathbb{C}[X]$.

It is good to study the ring of invariants! Why?

- We want to study set of orbits as a geometric object (notation: $X//G$). We hope that that this will be an affine variety. How to look for it?
- What should be regular functions on $X//G$? *Any* function on $X//G$ is a function on X that is constant on each orbit. So hope is that *regular* functions on $X//G = \mathbb{C}[X]^G$.
- Since the ring of regular functions on an affine variety characterizes that variety, we could *define* $X//G$ to be the affine variety associated to the ring of invariants.
- $X \to$ set of orbits as a map of affine varieties will then be associated to the inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$. We have to hold our horses. There are subtle issues, but for now ...

Finite generation of invariants

First question: is $\mathbb{C}[X]^G$ finitely generated as a ring? Then it will be a quotient of a polynomial ring by an ideal, which is generated by finitely many polynomials. We can do geometry with it!

Theorem: For a reductive G acting on affine X, the ring of invariants is finitely generated.

1. Reynolds operator R: we get a unique G-linear projection $R : \mathbb{C}[X] \to \mathbb{C}[X]^G$. We have this by complete reducibility on each finite dimensional G-stable subspace and they all patch to give a single operator R on all of $\mathbb{C}[X]$.
2. Observe that R commutes with multiplication by invariants i.e. it is $\mathbb{C}[X]^G$-linear.
3. Reduce to the case of a rational *linear* action by linearizing G-action on X as before.
4. Now the ring of functions $\mathbb{C}[x_1, \dots, x_m]$ is graded. There is a clever argument by induction on degree, using the fact that every *ideal* in a polynomial ring is finitely generated.

Thank you.