gct2022: School and Conference on Geometric Complexity Theory
2021-Aug-24

Geometric complexity theory

GCT1 and GCT2 and newer insights

Christian Ikenmeyer

UNIVERSITY OF LIVERPOOL

- The separation of complexity classes such as P and NP is one of the most fundamental open problems at the intersection of theoretical computer science and mathematics.

  Very few techniques are known that could potentially break the 1994 Razborov-Rudich natural proofs barrier.

- In 2001 Mulmuley and Sohoni published **[Geometric ComplexityTheory 1]** in which they describe a method that could potentially break the barrier.
  - It is built on Valiant's algebraic complexity theory framework (1979).
  - It is inspired by Mulmuley's geometric result that $P \neq NC$ *in an algebraic model* (1999).
  - It defines **border complexity**, which is defined independently by Bürgisser (2001)[1].
  - It proposes to prove border complexity lower bounds using representation theory, which is developed further in **[GCT2]** (2008).

[1] Bürgisser (2001) shows analogies to tensors and matrix multiplication, [Bini 1980] and [Lehmkuhl Lickteig 1989].

Determinantal Complexity

- Valiant (1979) found a close connection between complexity questions and natural questions in algebra:

Theorem (Valiant 1979)

Every multivariate polynomial h can be written as the determinant of a square matrix whose entries are polynomials of degree ≤ 1.

Example: $$h := y + 2x + xz + 2xy - x^2 z = \det \begin{pmatrix} x & y & 0 \\ -1 & z + y + 2 & x \\ 1 & z & 1 \end{pmatrix}$$

Def.: Required dimension of the matrix is called the **determinantal complexity** $\mathrm{dc}(h)$.

In the example we have $\mathrm{dc}(h) \leq 3$.

VBP = "easy to compute".

The class VBP is defined as the set of all sequences of polynomials h_m with polynomially bounded $\mathrm{dc}(h_m)$.

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Examples:

- $\det_m = \sum_{\pi \in \mathfrak{S}_m} \mathrm{sgn}(\pi) \prod_{i=1}^{m} x_{i,\pi(i)} = \det \begin{pmatrix} x_{1,1} & \cdots & x_{1.m} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m.m} \end{pmatrix} \in$ VBP, because $\mathrm{dc}(\det_m) = m$
- $x_1 x_2 \cdots x_m = \det \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m \end{pmatrix} \in$ VBP, because $\mathrm{dc}(x_1 \cdots x_m) = m$
- $x_1^m + x_2^m + \cdots + x_m^m \in$ VBP, because $\mathrm{dc}(x_1^m + x_2^m + \cdots + x_m^m) \leq m(m-1) + 1$

  For example $x_1^3 + x_2^3 + x_3^3 = \det \begin{pmatrix} 0 & 0 & x_1 & 0 & x_2 & 0 & x_3 \\ x_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 1 & 0 & 0 \\ x_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3 & 1 \end{pmatrix}$

The permanent polynomial and VNP

$\text{per}_m(x_{1,1}, x_{1,2}, \ldots, x_{m,m}) := \sum_{\pi \in \mathfrak{S}_m} x_{1,\pi(1)} x_{2,\pi(2)} \cdots x_{m,\pi(m)}$ (determinant, but without the signs)

- Set all $x_{i,j}$ to 0 or 1: then $\text{per}_m =$ number of perfect matchings in bipartite graph.
- Set all $x_{i,j}$ to 0 or 1: then $\text{per}_m =$ number of cycle covers in directed graph.
- Applications in theor. physics: Wavefunctions describing identical bosons
- #P-complete as a function

Valiant's universality theorem holds also for the permanent:

Every multivariate polynomial h can be written as the permanent of a matrix whose entries are polynomials of degree ≤ 1.

Def.: Required size of the matrix is called the **permanental complexity** $\text{pc}(h)$.
The class VNP consists of all sequences of polynomials h_m with polynomially bounded $\text{pc}(h_m)$.
Since $\text{pc}(\det_m)$ is polynomially bounded, it follows VBP $\subseteq$ VNP.

Valiant's "Determinant vs Permanent" Conjecture (1979)

- VBP $\neq$ VNP. Equivalently: $\text{dc}(\text{per}_m)$ is not polynomially bounded.

Remark: $\text{per}_m = \det_m$ over characteristic 2, but we can replace per_m by the Hamiltonian cycle polynomial.

Connections to Boolean complexity

Separating $\mathsf{VBP} \neq \mathsf{VNP}$ is "easier" than separating classes in Boolean complexity (Bürgisser 1998):

- $\mathsf{P}/\mathrm{poly} \neq \mathsf{NP}/\mathrm{poly}$ implies $\mathsf{VBP} \neq \mathsf{VNP}$ over finite fields.
- $\mathsf{P}/\mathrm{poly} \neq \mathsf{NP}/\mathrm{poly}$ implies $\mathsf{VBP} \neq \mathsf{VNP}$ over $\mathbb{C}$, assuming the generalized Riemann hypothesis.

[2]

$\mathsf{P}/\mathrm{poly} \neq \mathsf{NP}/\mathrm{poly}$ is widely believed: If $\mathsf{P}/\mathrm{poly} = \mathsf{NP}/\mathrm{poly}$, then

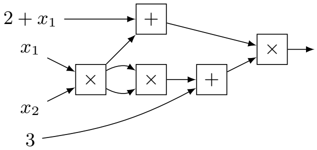
- $\mathsf{PH} = \Sigma_2^{\mathsf{P}}$ (Karp-Lipton, 1980, Sipser) and
- $\mathsf{AM} = \mathsf{MA}$ (Arvind, Köbler, Schöning, 1995).

[2] also holds for VF or VP instead of VBP, defined on next slides

Summary: Determinant vs permanent

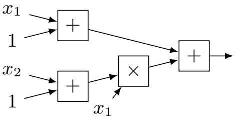
- If PH does not collapse and assuming GRH, then VBP $\neq$ VNP over $\mathbb{C}$.
- Valiant's model replaces Turing machines with determinants.

Algebraic Circuits and Formulas



- **Algebraic circuit**: Directed acyclic graph. Leaves labeled with degree 1 polynomials. Inner nodes (=gates) labeled with $+$ or $\times$.
- Every algebraic circuit computes a polynomial. Example on the left: $((x_1x_2)^2+3)\cdot((x_1x_2)+2+x_1) = x_1^3x_2^3 + x_1^3x_2^2 + 2x_1^2x_2^2 + 3x_1x_2 + 3x_1 + 6$
- Every polynomial can be computed by some algebraic circuit.

- size := number of gates
- The **algebraic circuit complexity** $L(h)$ is the size of a smallest algebraic circuit computing h.
- The class VP consists of all sequences of polynomials h_m with polynomially bounded $L(h_m)$ and polynomially bounded degree.
  Example: $(\det_m) \in$ VP.
- An **algebraic formula** is an algebraic circuit whose graph is a tree. For example:



Def.: The **algebraic formula complexity** $L_e(h)$ is defined as the size of a smallest algebraic formula computing h.
The class VF consists of all sequences of polynomials h_m with polynomially bounded $L_e(h_m)$.

Classes of efficient algebraic computation

$$\text{VF} \subseteq \text{VBP} \subseteq \text{VP}$$

VF $\subseteq$ VBP $\subseteq$ VP $\subseteq$ VNP

- The left three classes are classes of "efficient algebraic computation".
- The left three classes coincide if we allow quasipolynomial complexity instead of only polynomial complexity.

Common other names for these classes in the literature:

- $\mathsf{VF} = \mathsf{VP}_\mathsf{e}$
- $\mathsf{VBP} = \mathsf{VP}_\mathsf{s} = \mathsf{VP}_\mathsf{ws} = \mathsf{VDET}$

"Valiant's conjecture" can mean any one of the three:

- $\mathsf{VF} \neq \mathsf{VNP}$ (The permanent requires superpolynomially large formulas)
- $\mathsf{VBP} \neq \mathsf{VNP}$ (The permanent requires superpolynomially large determinants)
- $\mathsf{VP} \neq \mathsf{VNP}$ (The permanent requires superpolynomially large circuits)

Complete polynomials

$$h \ := \ y + 2x + xz + 2xy - x^2 z \ = \ \det\begin{pmatrix} x & y & 0 \\ -1 & z+y+2 & x \\ 1 & z & 1 \end{pmatrix}$$

This is an **evaluation** (also called projection) of the determinant polynomial.

$$\det_n \ := \ \det\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \ = \sum_{\pi \in \mathfrak{S}_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} x_{i,\pi(i)}$$

$h = \det_3(x, y, 0, -1, z+y+2, x, 1, z, 1)$

In the same way as determinantal complexity and permanental complexity:

- For a sequence c_n of polynomials we define the c-complexity of h as the smallest n such that h is a projection of c_n.
- We obtain a corresponding complexity class: Sequences of polynomials with polynomially bounded c-complexity.
- We say that c_n is **complete** for that class.

$\det_n$ is VBP-complete.
per_n is VNP-complete.

Models of computation with affine linear inputs

- The iterated 3×3 matrix multiplication (degree n on $9n$ variables) is VF-complete:

$$\mathrm{imm}_3^{(n)} := \left(\begin{pmatrix} x_{1,1}^{(1)} & x_{1,2}^{(1)} & x_{1,3}^{(1)} \\ x_{2,1}^{(1)} & x_{2,2}^{(1)} & x_{2,3}^{(1)} \\ x_{3,1}^{(1)} & x_{3,2}^{(1)} & x_{3,3}^{(1)} \end{pmatrix} \cdots \begin{pmatrix} x_{1,1}^{(n)} & x_{1,2}^{(n)} & x_{1,3}^{(n)} \\ x_{2,1}^{(n)} & x_{2,2}^{(n)} & x_{2,3}^{(n)} \\ x_{3,1}^{(n)} & x_{3,2}^{(n)} & x_{3,3}^{(n)} \end{pmatrix} \right)_{1,1}$$

  Instead of the (1,1) entry, one can use the **trace** of the matrix.

- VBP has several complete polynomials
  - $\det_n$ or
  - $\mathrm{imm}_n^{(n)}$ ((1,1) entry or trace) or
  - or

$$\mathrm{trace}\left(\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}^n \right)$$

  (or the (1,1) entry instead of the trace)

- VP can also be defined via complete polynomials, a graph homomorphism polynomial [Durand Mahajan Malod de Rugy-Altherre Saurabh 2014], or via stack algebraic branching programs [Mengel 2013, Chaugule Limaye Pandey 2020]

Models of computation with **homogeneous** linear inputs

Homogeneous iterated matrix multiplication complexity [1]

Let h be a homogeneous degree d polynomial. The **homogeneous iterated matrix multiplication complexity** $\mathrm{himmc}(h)$ (or **homogeneous algebraic branching program width**) is defined as the smallest n such that there exist **homogeneous** linear polynomials $\ell_{i,j}^{(k)}$ with

$$h = \left(\begin{pmatrix} \ell_{1,1}^{(1)} & \cdots & \ell_{1,n}^{(1)} \\ \vdots & \ddots & \vdots \\ \ell_{n,1}^{(1)} & \cdots & \ell_{n,n}^{(1)} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,1}^{(d)} & \cdots & \ell_{1,n}^{(d)} \\ \vdots & \ddots & \vdots \\ \ell_{n,1}^{(d)} & \cdots & \ell_{n,n}^{(d)} \end{pmatrix} \right)_{1,1}$$

Theorem (homogenization of VBP)

A sequence h of homogeneous polynomials with polyn. bd. degree is in VBP iff $\mathrm{himmc}(h)$ is polyn. bounded.

[Grenet 2012] proved that $\mathrm{himmc}(\mathrm{per}_m) \leq \binom{m}{\lceil m/2 \rceil}$.

Reformulization of Valiant's VBP $\neq$ VNP conjecture

$\mathrm{himmc}(\mathrm{per}_m)$ is not polynomially bounded.

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[1] One could also use the trace instead of (1,1), but we know that in the noncommutative setting this makes things mildy more complicated [Bläser, I, Mahajan, Pandey, Saurabh 2020].
Also possible (but polynomial is not defined by its symmetry group): Trace of matrix power or (1,1) entry of matrix power.

Several advantages of homogeneous settings:

- conceptually simpler transition to geometric complexity theory (no “padding”)
- not as many “peculiarities”:
  - no “useless” types of representations as in [Kadish, Landsberg 2014]
  - no strong occurrence obstruction no-go results as in [Bürgisser, I, Panova 2019] (although even in homogeneous settings multiplicity obstructions are stronger than occurrence obstructions [Dörfler, I, Panova 2019])
  - No surprising computability results as in [Bringmann-I-Zuiddam 2018] or [Kumar 2020]

A classical homogeneous setting: Waring rank

Classically studied in algebraic geometry: **homogeneous** $\Sigma\Lambda\Sigma$**-circuits**. Is a useful testbed for some GCT ideas. Every homogeneous degree d polynomial h can be written as a sum of d-th powers of homogenoeous linear polynomials ℓ_i:

$$h = \sum_{i=1}^{r} (\ell_i)^d.$$

The smallest r possible is called the Waring rank $\mathrm{WR}(h)$ of h.

Example:
$6x^2y = (x+y)^3 + (y-x)^3 - 2y^3$, hence $\mathrm{WR}(x^2y) \le 3$. In fact, $\mathrm{WR}(x^2y) = 3$.

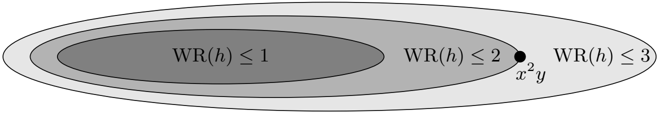
Summary: Other algebraic models of computation

- Valiant's conjecture VBP $\neq$ VNP can be phrased homogeneously: $\mathrm{himmc}(\mathrm{per}_m)$ grows superpolynomially
- The non-homogeneous settings have some peculiarities
- Homogeneous $\Sigma\Lambda\Sigma$-circuits are a classical area of study in algebraic geometry (Waring rank)

1. Determinant vs permanent
2. Homogeneous algebraic models of computation
3. Border complexity
4. Algebraic Geometry
5. Representation theory

Example:

$h = x^2y = \frac{1}{6}((x+y)^3 + (y-x)^3 - 2y^3)$, $\mathrm{WR}(h) = 3$.



$$\frac{1}{3\varepsilon}\left((x+\varepsilon y)^3 - x^3\right) = x^2y + \varepsilon xy^2 + \frac{\varepsilon^2}{3}y^3 \xrightarrow{\varepsilon \to 0} x^2y$$

This makes determining $\mathrm{WR}(x^2y)$ subtle!

If a continuous function f vanishes on all h with $\mathrm{WR}(h) \le 2$, then f also vanishes on x^2y.

Definition (border Waring rank)

The **border Waring rank** $\underline{\mathrm{WR}}(h)$ is defined as the smallest n such that h can be approximated arbitrarily closely by polynomials of Waring rank $\le n$. (e.g., $\underline{\mathrm{WR}}(x^2y) = 2$)

Main advantage: $X_n := \{h \mid \underline{\mathrm{WR}}(h) \le n\} = \overline{\{h \mid \mathrm{WR}(h) \le n\}}$ is **closed**, so it is guaranteed that non-membership $p \notin X_n$ can be proved by finding a continuous function f that vanishes on X_n, but does not vanish on p.

Example: Consider the 3-dim vector space $\mathbb{C}[x,y]_2$. Let

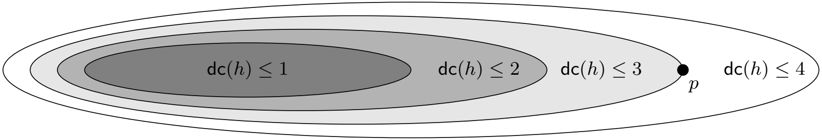
$X_1 := \{h \mid \underline{\mathrm{WR}}(h) \le 1\} = \{h \mid \exists \alpha, \beta \in \mathbb{C} : h = (\alpha x + \beta y)^2\} = \{ax^2 + bxy + cy^2 \mid b^2 - 4ac = 0\}$

A lower bound: $\underline{\mathrm{WR}}(xy) \ge 2$, because $1^2 - 4 \cdot 0 \cdot 0 = 1 \ne 0$.

- Such $f = b^2 - 4ac$ is sometimes called a **polynomial obstruction** or a **separating polynomial**.

Approximations also help in most other computational models.

If c is a complexity measure, then we can define the border complexity measure $\underline{c}$ via:
$\underline{c}(h)$ is the smallest n such that h can be approximated arbitrarily closely by polynomials h_ε with $c(h_\varepsilon) \leq n$.



- Landsberg (further studied by Hüttenhain and Lairez):
  For $p = x_1^2 y_1 + x_2^2 y_2 + x_3^2 y_3 + x_1 x_2 z_3 + x_1 z_2 x_3 + z_1 x_2 x_3$ we have $\underline{\text{dc}}(p) = 3 < \text{dc}(p)$.
- [Allender Wang 2016] and [Bringmann I Zuiddam 2018]: For $h = x_0^{39}(x_1x_2 + x_3x_4 + \cdots + x_{15}x_{16})$ we have $\underline{\text{himmc}}(h) = 2 < \text{himmc}(h)$.
- Trace instead of (1,1)-entry:

$$\text{tr}\left(\begin{pmatrix} x_1 & \frac{1}{\varepsilon}x_2 \\ \frac{1}{\varepsilon}x_3 & x_4 \end{pmatrix}\begin{pmatrix} \varepsilon x_5 & x_6 \\ x_7 & \varepsilon x_8 \end{pmatrix}\begin{pmatrix} \varepsilon x_9 & x_{10} \\ x_{11} & \varepsilon x_{12} \end{pmatrix}\right) \xrightarrow{\varepsilon \to 0} x_3x_6x_{12} + x_2x_7x_9 + (x_3x_5 + x_4x_7)x_{10} + (x_1x_6 + x_2x_8)x_{11}$$

cannot be written as a trace of a prod. of three 2×2 matrices, but as a trace of a prod. of 3 larger matrices.

The main reason of doing this:

For all these measures, each set $\{h \mid \underline{c}(h) \leq n\}$ is closed.

We will actually see that each set is an **algebraic variety with an action of the general linear group**.

If c is a complexity measure, then we can define the border complexity measure $\underline{c}$ via:
$\underline{c}(h)$ is the smallest n such that h can be approximated arbitrarily closely by polynomials h_ε with $c(h_\varepsilon) \le n$.

$\mathsf{VF} = \{$polynomially bounded formula complexity$\}$	$\overline{\mathsf{VF}} = \{$polynomially bounded **border** formula complexity$\}$
$\mathsf{VBP} = \{$polynomially bounded dc$\}$	$\overline{\mathsf{VBP}} = \{$polynomially bounded $\underline{\mathrm{dc}}\}$
$\mathsf{VP} = \{$polynomially bounded circuit complexity$\}$	$\overline{\mathsf{VP}} = \{$polynomially bounded **border** circuit complexity$\}$

Strenghtening of Valiant's conjecture (Mulmuley Sohoni 2001, Bürgisser 2001)

$\mathsf{VNP} \not\subseteq \overline{\mathsf{VF}}, \quad \mathsf{VNP} \not\subseteq \overline{\mathsf{VBP}}, \quad \mathsf{VNP} \not\subseteq \overline{\mathsf{VP}}.$

The conj. "$\mathsf{VNP} \not\subseteq \overline{\mathsf{VBP}}$" is equivalent to the conj. "$\underline{\mathrm{himmc}}(\mathrm{per}_m)$ *is not polynomially bounded.*"

Strenghtening of Valiant's conjecture (Mulmuley Sohoni 2001, Bürgisser 2001)

$\mathsf{VNP} \not\subseteq \overline{\mathsf{VF}}, \quad \mathsf{VNP} \not\subseteq \overline{\mathsf{VBP}}, \quad \mathsf{VNP} \not\subseteq \overline{\mathsf{VP}}.$

Major open problems ([Mulmuley Sohoni 2001] and [Bürgisser 2001]):

$$\mathsf{VF} \stackrel{?}{=} \overline{\mathsf{VF}}, \quad \mathsf{VBP} \stackrel{?}{=} \overline{\mathsf{VBP}}, \quad \mathsf{VP} \stackrel{?}{=} \overline{\mathsf{VP}},$$

because this would mean that the strenghtened conj. is equivalent to Valiant's original conj., so we could work with the closed sets (which are algebraic varieties with a group action).
We do not even know if $\overline{\mathsf{VF}} \subseteq \mathsf{VNP}$.

"Toy model": Let VWaring be the set of sequences of polynomials whose Waring rank is polynomially bounded.
Open question: $\overline{\mathsf{VWaring}} \stackrel{?}{=} \mathsf{VWaring}$.

Results:

- [Nisan 1991] showed that himmc $=$ $\underline{\text{himmc}}$ is the noncommutative world, hence $\mathsf{VBP}_{\text{non-comm}} = \overline{\mathsf{VBP}_{\text{non-comm}}}$.
- [Bläser Dörfler I 2020] prove $\underline{\mathrm{WR}}(h) \leq \text{himmc}(h)$. Hence $\overline{\mathsf{VWaring}} \subseteq \mathsf{VBP}$ (which was discovered earlier by Forbes via ROABPs).
- [Dutta Dwivedi Saxena 2021] prove $\overline{\Sigma^k \Pi \Sigma} \subseteq \mathsf{VBP}$

Many recent papers study the power and limitations of border complexity, but it is still a mystery (Bläser, Bringmann, Dutta, Dwivedi, Forbes, Grochow, Kumar, Mahajan, Medini, Mulmuley, Pandey, Sanyal, Saxena, Saurabh, Shpilka, Quiao, Zuiddam).

Summary: Algebraic Models of Computation and Border Complexity

- The conj. "VNP $\not\subseteq \overline{\text{VBP}}$" is equivalent to the conj. "$\underline{\text{himmc}}(\text{per}_m)$ *is not polynomially bounded.*"
- The set $\{h \mid \underline{\text{himmc}}(h) \leq n\}$ is closed, so proving $\underline{\text{himmc}}(\text{per}_m) > n$ can be done with continuous functions f that vanish on $\{h \mid \underline{\text{himmc}}(h) \leq n\}$ and not on per_m.

We will now see that these closed sets are actually algebraic varieties.
This will allow us to impose more restrictions on f.

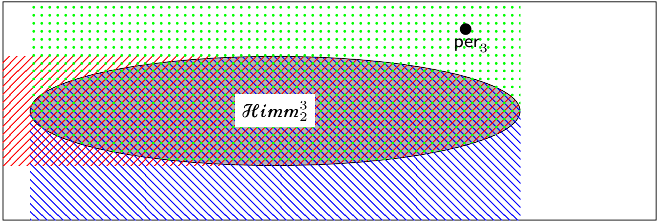
$X \subseteq \mathbb{C}^\eta$ is called an **algebraic variety** if there exist polynomials $f_1, \dots, f_N$ such that

$$X = \{h \mid f_1(h) = f_2(h) = \cdots = f_N(h) = 0\}.$$

Example ($\eta = 3$): $\{h \in \mathbb{C}[x,y]_2 \mid \underline{\mathrm{WR}}(h) \le 1\} = \{ax^2 + bxy + cy^2 \mid \underbrace{b^2 - 4ac}_{=:f_1} = 0\}$

Polynomial obstructions must exist

If $X_n = \{h \mid \underline{c}(h) \le n\}$ is an algebraic variety and $p \notin X_n$, then there exists a polynomial f with $f(X_n) = \{0\}$ and $f(p) \neq 0$.



$\mathcal{H}imm_2^3 = \{h \text{ hom deg 3} \mid \underline{\text{himmc}}(h) \le 2\}$

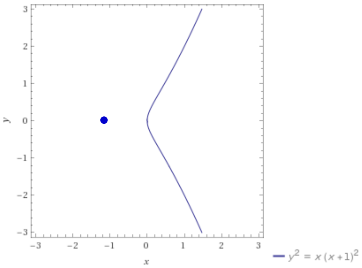
Remark: The method of partial derivatives, shifted partial derivatives, and any "rank based method" prove the existence of polynomial obstructions: determinants of minors of certain matrices.

Meta-Theorem (over $\mathbb{C}$)

Let c be an algeb. compl. measure given by a syntactic circuit class with homogeneous linear inputs. Then $\{h \mid \underline{c}(h) \leq n\}$ is an algebraic variety.

Proof approach:

- First observation: $\{h \mid \underline{c}(h) \leq n\} = \overline{\{h \mid c(h) \leq n\}}$
- The **Zariski closure** $\overline{Y}^Z$ of any subset $Y \subseteq \mathbb{C}^n$ is the smallest algebraic variety containing Y. In general $Y \subseteq \overline{Y} \subseteq \overline{Y}^Z$. We want $\overline{Y} = \overline{Y}^Z$, because then $\overline{\{h \mid c(h) \leq n\}} = \overline{\{h \mid c(h) \leq n\}}^Z$ is an algebraic variety. This is not always true over $\mathbb{R}$:



The way around this problem over $\mathbb{C}$ can be concisely stated in the language of **orbit closures**.

Orbit closures

$$x^2y = \lim_{\varepsilon\to 0}\tfrac{1}{3\varepsilon}\Big((x+\varepsilon y)^3 - x^3\Big) = \lim_{\varepsilon\to 0}\Big((s_\varepsilon x + \varepsilon s_\varepsilon y)^3 + (\omega s_\varepsilon x)^3\Big)$$

with $s_\varepsilon^3 = \frac{1}{3\varepsilon}$ and $\omega^3 = -1$.

$$\begin{aligned}(s_\varepsilon x + \varepsilon s_\varepsilon y)^3 + (\omega s_\varepsilon x)^3 \quad &= \quad (x^3+y^3)(s_\varepsilon x + \varepsilon s_\varepsilon y,\ \omega s_\varepsilon x) \ = \ (x^3+y^3)\left(\begin{pmatrix} x & y\end{pmatrix}\begin{pmatrix} s_\varepsilon & \omega s_\varepsilon \\ \varepsilon s_\varepsilon & 0\end{pmatrix}\right) \\ &=: \quad \begin{pmatrix} s_\varepsilon & \omega s_\varepsilon \\ \varepsilon s_\varepsilon & 0\end{pmatrix}\cdot(x^3+y^3)\end{aligned}$$

$\mathbb{C}^{2\times 2}(x^3+y^3) := \{g(x^3+y^3) \mid g \in \mathbb{C}^{2\times 2}\} = \{h \in \mathbb{C}[x,y]_2 \mid \mathrm{WR}(h) \le 2\}$ is a **monoid orbit**.

- $x^2y \notin \mathbb{C}^{2\times 2}(x^3+y^3)$
- $x^2y \in \overline{\mathbb{C}^{2\times 2}(x^3+y^3)} = \{h \in \mathbb{C}[x,y]_2 \mid \underline{\mathrm{WR}}(h) \le 2\}$
- Since $\mathsf{GL}_2 \subset \mathbb{C}^{2\times 2}$ is dense: $\overline{\mathbb{C}^{2\times 2}(x^3+y^3)} = \overline{\mathsf{GL}_2(x^3+y^3)}$. This is called a **group orbit closure**.
- Formally, $(gp)(x) = p(xg)$.

For complexity classes given by syntactic circuit class with homogeneous linear inputs: $\{h \mid \underline{c}(h) \le n\}$ is a linear space intersected with a finite union of orbit closures.

Meta-Theorem (over $\mathbb{C}$)

Let c be an algeb. compl. measure given by a syntactic circuit class. Then $\{h \mid \underline{c}(h) \leq n\}$ is an algebraic variety.

Proof sketch:

- First observation: $\{h \mid \underline{c}(h) \leq n\} = \overline{\{h \mid c(h) \leq n\}}$
- The **Zariski closure** $\overline{Y}^Z$ of any subset $Y \subseteq \mathbb{C}^n$ is the smallest algebraic variety containing Y. In general $Y \subseteq \overline{Y} \subseteq \overline{Y}^Z$. We want $\overline{Y} = \overline{Y}^Z$, because then $\overline{\{h \mid c(h) \leq n\}} = \overline{\{h \mid c(h) \leq n\}}^Z$ is an algebraic variety.
- A set is **constructible** if it is a finite union/intersection/complement of algebraic varieties.
- Chevalley's theorem implies: Orbit closures are constructible.
- For constructible sets, Euclidean and Zariski closures coincide (over $\mathbb{C}$) $\square$

Fundamental Conclusion: All border complexity lower bounds can be proved via polynomials

All lower bounds on any border complexity measure (defined by a syntactic circuit class)

- can be phrased as $p \notin X$ for some algebraic variety X
- can be proved by a polynomial f with $f(X) = \{0\}$ and $f(p) \neq 0$. (analogous to the discriminant)

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Conclusion: All lower bounds on any border complexity measure

- can be phrased as $p \notin X$ for some algebraic variety X
- can be proved by a polynomial f with $f(X) = \{0\}$ and $f(p) \neq 0$. (analogous to the discriminant)

What is the complexity of these f? This is a question about **algebraic natural proofs**.

Several recent results: [Forbes, Shpilka, Volk 2017], [Grochow, Kumar, Saks, Saraf 2017], [Bläser, I, Jindal, Lysikov 2018], [Bläser, I, Lysikov, Pandey, Schreyer 2019], [Chatterjee, Kumar, Ramya, Saptharishi, Tengse 2021], [Kumar, Ramya, Saptharishi, Tengse 2021]

Mulmuley and Sohoni:
We should use the fact that $\{h \mid \underline{c}(h) \leq n\}$ is not only an algebraic variety, but also carries an

action of the general linear group.

Summary: Algebraic geometry

- $\{h \in \mathbb{C}[x_1, \ldots, x_{n^2 m}]_m \mid \underline{\text{himmc}}(h) \leq n\} = \overline{\mathsf{GL}_{n^2 m} \mathsf{imm}_n^{(m)}}$ is an orbit closure, in particular an algebraic variety.
- $\underline{\text{himmc}}(\mathsf{per}_m) > n$ iff $\mathsf{per}_m \notin \overline{\mathsf{GL}_{n^2 m} \mathsf{imm}_n^{(m)}}$
- $\underline{\text{himmc}}(\mathsf{per}_m) > n$ iff there exists a polynomial f with $f(\overline{\mathsf{GL}_{n^2 m} \mathsf{imm}_n^{(m)}}) = \{0\}$ and $f(\mathsf{per}_m) \neq 0$.
- We want to study these f using representation theory.

Consider what happens in the 3-dimensional vector space $\mathbb{C}[x,y]_2$ when we exchange the roles of x and y.

- This operation is a linear endomorphism $\tau : \mathbb{C}[x,y]_2 \to \mathbb{C}[x,y]_2,\ \tau(p) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p$

  We say that $\mathbb{C}[x,y]_2$ is a **representation of** $\mathfrak{S}_2$.

A pair $(\mathscr{V}, \varrho)$ of a fin. dim. vector space $\mathscr{V}$ and a group homom. $\varrho : G \to \mathsf{GL}(\mathscr{V})$ is called a **representation**.

$\tau(xy) = xy$ is an **invariant**. $\tau(x^2+y^2) = x^2+y^2$ is an **invariant**. $\tau(x^2-y^2) = y^2-x^2$ is a **skew-invariant**.

$$\mathbb{C}[x,y]_2 = \underbrace{\langle xy, x^2+y^2 \rangle}_{=:\mathbb{C}[x,y]_2^{\mathrm{inv}}} \oplus \underbrace{\langle x^2-y^2 \rangle}_{=:\mathbb{C}[x,y]_2^{\mathrm{skew\text{-}inv}}} .$$

A **subrepresentation** of a representation is a linear subspace that is closed under the group action.

We write $\mathsf{mult}_{\mathrm{inv}}(\mathbb{C}[x,y]_2) = \dim \mathbb{C}[x,y]_2^{\mathrm{inv}} = 2$ $\qquad \mathsf{mult}_{\mathrm{skew\text{-}inv}}(\mathbb{C}[x,y]_2) = \dim \mathbb{C}[x,y]_2^{\mathrm{skew\text{-}inv}} = 1$

Let $W := \mathbb{C}[x,y]_2$ with basis $\{x^2, xy, y^2\}$. Rename the basis vectors $a := x^2$, $b := xy$, $c := y^2$.

Let $\mathbb{C}[W]_2$ with basis $\{a^2, ab, ac, b^2, bc, c^2\}$. $\qquad (\tau f)(p) := f(\tau p)$

$$\tau(a) = c,\ \tau(b) = b,\ \tau(c) = a. \qquad \tau(a^2) = c^2,\ \tau(ab) = bc,\ \ldots,\ \tau(c^2) = a^2.$$

$$\mathbb{C}[W]_2 = \underbrace{\langle ac, b^2, a^2+c^2, ab+bc \rangle}_{\text{isotypic component of invariants}} \oplus \underbrace{\langle a^2-c^2, ab-bc \rangle}_{\text{isotypic component of skew-invariants}}$$

$\mathsf{mult}_{\mathsf{inv}}(\mathbb{C}[W]_2) := \dim\{f \mid \tau f = f\} = 4$ $\qquad \mathsf{mult}_{\mathsf{skew\text{-}inv}}(\mathbb{C}[W]_2) := \dim\{f \mid \tau f = -f\} = 2$

Note that $b^2 - 4ac$ lies in an isotypic component.

Let $\mathbb{C}[W]_2$ with basis $\{a^2, ab, ac, b^2, bc, c^2\}$.

- On the previous slide: $\tau : \mathbb{C}[x,y]_2 \to \mathbb{C}[x,y]_2,\ \tau(p) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p$
- Now: Instead of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we can use every $g \in \mathsf{GL}_2$ (formally, gf is defined via $(gf)(p) := f(g^t p)$)
- For example: $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x^2 = (x+y)^2 = x^2 + 2xy + y^2$ $\quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} a = a + 2b + c$
  - $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} a^2 = (a+2b+c)^2 = a^2 + 4ab + 2ac + 4b^2 + 4bc + c^2$
  - $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} a^2 = a^2$
  - $g(b^2 - 4ac) = (\det(g))^2 (b^2 - 4ac)$

$$\mathbb{C}[W]_2 = \underbrace{\langle b^2 - 4ac \rangle}_{\text{irreducible}} \quad \oplus \quad \underbrace{\langle a^2, ab, b^2 + 4ac, bc, c^2 \rangle}_{\text{irreducible}}$$

A representation $\mathscr{V}$ is called **irreducible** if 0 and $\mathscr{V}$ are the only subrepresentations.

Theorem: GL_N is reductive

Every GL_N-representation can be decomposed into a direct sum of irreducibles.

The vanishing ideal

Theorem: GL_N is reductive

Every GL_N-representation can be decomposed into a direct sum of irreducibles.

Recall

- $W = \mathbb{C}[x, y]_2$
- $\mathbb{C}[W] = \mathbb{C}[a, b, c]$.

Let $X \subseteq W$ be a Zariski-closed cone, for example the set of polynomials of Waring rank ≤ 1.

Vanishing ideal

The ideal $I(X) = \{f \in \mathbb{C}[W] \mid f(X) = \{0\}\}$ is called the **vanishing ideal** of X. Define $I(X)_d := I(X) \cap \mathbb{C}[W]_d$.

$I(X)_d \subseteq \mathbb{C}[W]$ is a subrepresentation (because X is an orbit closure)!
Example:

$$\mathbb{C}[W]_2 = \langle b^2 - 4ac\rangle \oplus \langle a^2, ab, b^2 + 4ac, bc, c^2\rangle \qquad I(X)_2 = \langle b^2 - 4ac\rangle$$

Conclusion (decompose the vanishing ideal into irreducibles)

All lower bounds on any border complexity measure with hom. lin. inputs can be proved by a polynomial f in an irreducible subrepresentation of $I(X)$ (in particular $f(X) = \{0\}$), and $f(p) \neq 0$.

Highest weight vectors

Observation: $g(b^2 - 4ac) = \det(g)^2(b^2 - 4ac)$

Theorem (highest weight vectors)

Every irreducible GL_N-representation contains (up to scale) exactly one **highest weight vector** (HWV):

- $\mathrm{diag}(\alpha_1, \dots, \alpha_N) f = \alpha_1^{\lambda_1} \cdots \alpha_N^{\lambda_N} f$
- $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} f = f$

The exponent vector λ is called the **weight** of f.
Two irreducible GL_N-representations are called **isomorphic** if the weights of their HWVs coincide.

$b^2 - 4ac$ is a HWV of weight $(2,2)$. a^2 is a HWV of weight $(4,0)$.

$$\mathbb{C}[W]_2 = \underbrace{\langle b^2 - 4ac \rangle}_{\text{irreducible of type (2,2)}} \oplus \underbrace{\langle a^2, ab, b^2 + 4ac, bc, c^2 \rangle}_{\text{irreducible of type (4,0)}} \qquad I(X)_2 = \underbrace{\langle b^2 - 4ac \rangle}_{\text{irreducible of type (2,2)}}$$

Conclusion

If $h \notin X$, then there exists a HWV f such that gf vanishes on X and $(gf)(h) \neq 0$ for a generic matrix g.
This means, all border complexity lower bounds with homog. linear inputs can be proved via HWVs.

Mulmuley and Sohoni propose to circumvent HWVs:

Theorem

The number of isomorphic copies of irreducibles in a decomposition is **independent of the decomposition**. This number is called the **multiplicity**.

$$\mathbb{C}[W]_2 = \underbrace{\langle b^2 - 4ac\rangle}_{\text{irreducible of type (2,2)}} \oplus \underbrace{\langle a^2, ab, b^2+4ac, bc, c^2\rangle}_{\text{irreducible of type(4,0)}} \qquad I(X)_2 = \underbrace{\langle b^2 - 4ac\rangle}_{\text{irreducible of type (2,2)}}$$

$\mathsf{mult}_{(2,2)}(\mathbb{C}[W]_2) = 1$ $\mathsf{mult}_{(4,0)}(\mathbb{C}[W]_2) = 1$ $\mathsf{mult}_{(3,1)}(\mathbb{C}[W]_2) = 0$ $\mathsf{mult}_{(2,2)}(I(X)_2) = 1$ $\mathsf{mult}_{(4,0)}(I(X)_2) = 0$ [3]

Fix n, m. Let $v = \max(m^2, 2n + n^2(m-2))$. We want to show $\mathsf{per}_m \notin \overline{\mathsf{GL}_v \mathsf{imm}_n^{(m)}}$.

$$h \in \overline{\mathsf{GL}_v \mathsf{imm}_n^{(m)}} \iff \overline{\mathsf{GL}_v h} \subseteq \overline{\mathsf{GL}_v \mathsf{imm}_n^{(m)}}$$

Multiplicity obstruction

If there exists λ with $\mathsf{mult}_\lambda(I(\overline{\mathsf{GL}_v \mathsf{imm}_n^{(m)}})_d) > \mathsf{mult}_\lambda(I(\overline{\mathsf{GL}_v \mathsf{per}_m})_d)$, then there exists a HWV proving $\mathsf{per}_m \notin \overline{\mathsf{GL}_v \mathsf{imm}_n^{(m)}}$. We call such λ **multiplicity obstructions**.

Major open question: Are multiplicity obstructions fine enough to prove strong lower bounds?

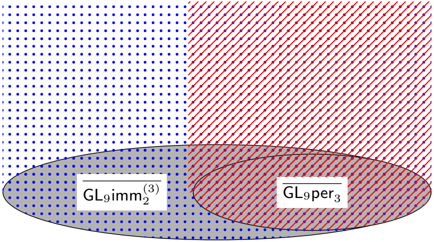
[3] The multiplicities in $\mathbb{C}[W]$ are called **plethysm coefficients**. Vectors λ with positive plethysm coefficient give points in a **moment cone**.

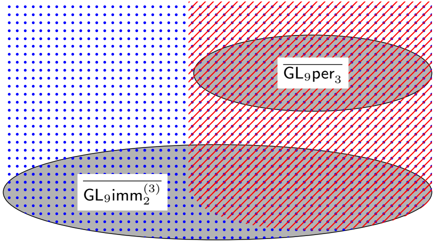
Multiplicity obstruction

If $\text{mult}_\lambda(I(\overline{\text{GL}_v \text{imm}_n^{(m)}})_d) > \text{mult}_\lambda(I(\overline{\text{GL}_v \text{per}_m})_d)$, then $\overline{\text{GL}_9 \text{per}_3} \not\subseteq \overline{\text{GL}_v \text{imm}_n^{(m)}}$.

It could be that $\overline{\text{GL}_9 \text{per}_3} \not\subseteq \overline{\text{GL}_9 \text{imm}_2^{(3)}}$, but $\forall \lambda$ we have $\text{mult}_\lambda(I(\overline{\text{GL}_v \text{imm}_n^{(m)}})_d) \leq \text{mult}_\lambda(I(\overline{\text{GL}_v \text{per}_m})_d)$.

These two situations look the same from the viewpoint of multiplicities:





$\text{mult}_\lambda(I(\overline{\text{GL}_9 \text{imm}_2^{(3)}})_d) = 1 \leq 2 = \text{mult}_\lambda(I(\overline{\text{GL}_9 \text{per}_3})_d)$.

This problem cannot occur for λ for which **all** HWVs of weight λ vanish on $\overline{\text{GL}_v \text{imm}_n^{(m)}}$. In other words:

$$\text{mult}_\lambda(\mathbb{C}[W])_d) = \text{mult}_\lambda(I(\overline{\text{GL}_v \text{imm}_n^{(m)}})_d) > \text{mult}_\lambda(I(\overline{\text{GL}_v \text{per}_m})_d)$$

Such λ are called **occurrence obstructions**. These are a special case of multiplicity obstructions.

- The occurrence obstruction approach for determinant vs permanent does not give superpolynomial lower bounds [Bürgisser I Panova 2016]. But that is not a homogeneous setting.
- Occurrence obstructions can prove lower bounds on the border rank of the matrix multiplication tensor [Bürgisser I 2011, 2013].
- There are settings in which multiplicity obstructions are provably stronger than occurrence obstructions [Dörfler, I, Panova 2019].
- Hope that multiplicities might be fine enough comes from [Larsen Pink 1990] and [Yu 2016].
- There are situations where multiplicity obstructions can be constructed just from the symmetry groups (=stabilizers) of the two points [I-Kandasamy 2018]

Coordinate ring

$\mathbb{C}[X] = \mathbb{C}[W]/I(X)$ and in each degree d: $\mathbb{C}[X]_d = \mathbb{C}[W]_d/I(X)_d$.
For all λ: $\mathsf{mult}_\lambda(\mathbb{C}[X]_d) = \mathsf{mult}_\lambda(\mathbb{C}[W]_d) - \mathsf{mult}_\lambda(I(X)_d)$

In many cases the coordinate ring of an orbit closure $\mathbb{C}[\overline{\mathsf{GL}_v p}]$ has a localization that can be studied via the algebraic Peter-Weyl theorem and representation theoretic branching rules:

$$\mathbb{C}[\overline{\mathsf{GL}_v p}]_\Phi = \mathbb{C}[\mathsf{GL}_v p] = \mathbb{C}[\mathsf{GL}_v/\mathsf{stab}(p)] = \mathbb{C}[\mathsf{GL}_v]^{\mathsf{stab}(p)\text{-inv}} = \bigoplus_\lambda S_{\lambda^*} \otimes S_\lambda^{\mathsf{stab}(p)\text{-inv}}$$

Here Φ is the so-called **fundamental invariant** [Bürgisser I 2017] and S_λ is the irreducible repr. of type λ.
Hence determining the multiplicity reduces to determining the dimension of an invariant space:
$\mathsf{mult}_{\lambda^*}\mathbb{C}[\overline{\mathsf{GL}_v p}]_\Phi = \dim(S_\lambda^{\mathsf{stab}(p)\text{-inv}})$

Summary

- There are several definitions of algebraic complexity, some affine and some homogeneous. The VBP vs VNP question can naturally be phrased in a homogeneous way via $\text{himmc}(\text{per}_m)$. This simplifies the transition to GCT.
- When studying border complexity over $\mathbb{C}$ a rich set of tools from algebraic geometry and representation theory becomes available. It also raises questions:
  - Is $\text{VBP} = \overline{\text{VBP}}$? In other words, are himmc and $\underline{\text{himmc}}$ polynomially related?
  - $\text{VNP} \not\subseteq \overline{\text{VBP}}$ is a question about the containment of orbit closures.
- All border complexity (with hom. lin. input) lower bounds can be proved via HWVs.
- Multiplicity obstructions can be used to circumvent having to compute HWVs. They can sometimes be obtained from the symmetry groups of the two points.
- Multiplicity obstructions are provably stronger than occurrence obstructions, but even occurrence obstructions work in some cases.

Thank you!