ORBIT CLOSURES IN GCT

TECHNIQUES FROM PROJECTIVE GEOMETRY

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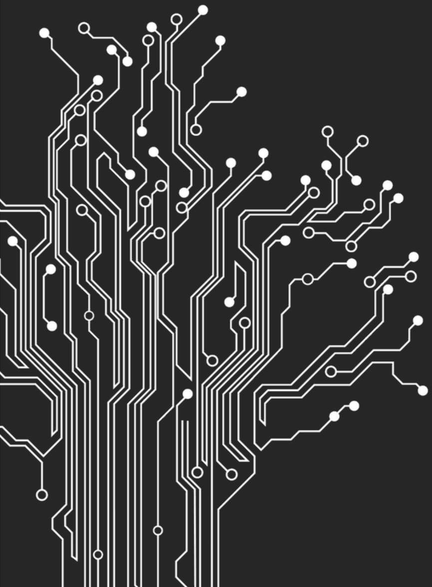


1. Introduction And Notation
2. The Rational Orbit Map
3. Maximal Linear Subspaces
4. Strategy And Reading Suggestions

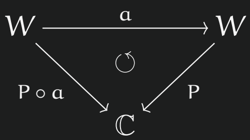
INTRODUCTION AND NOTATION

POLYNOMIALS AND ORBIT MAPS

THE SCENARIO

- $W := \mathbb{C}^n$
- $P \in \mathbb{C}[W] = \mathbb{C}[X_1, \ldots, X_n]$
- $a \in \mathrm{GL}(W) \subseteq \mathrm{End}(W)$

COMPOSITION AS RIGHT ACTION



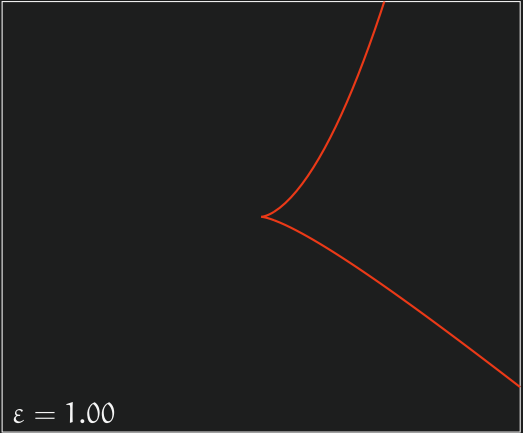
ORBIT & BOUNDARY

- $\mathrm{GL}(W)$ acts on $\mathbb{C}[W]_d$
- $\Omega_P := P \circ \mathrm{GL}(W) \subseteq P \circ \mathrm{End}(W)$
- $\partial\Omega_P := \overline{\Omega}_P \setminus \Omega_P$

GOAL

- We would like to understand $\partial\Omega_P$.
- $P \circ \mathrm{End}(W) \neq \overline{\Omega}_P$ in general.
- Something is missing!

LINEAR APPROXIMATIONS OF POLYNOMIALS



EXAMPLE

Polynomials $P, Q \in \mathbb{C}[X, Y, Z]$:

$$P := X \cdot Y^2 - Z^3$$

$$Q := Y \cdot (YZ - X^2)$$

$$a_\varepsilon := \frac{1}{3} \begin{pmatrix} 9\varepsilon^3 & 1 & 27\varepsilon^6 \\ 0 & -\varepsilon^{-3} & 0 \\ 3\varepsilon & \varepsilon^{-2} & 0 \end{pmatrix}$$

$$P \circ a_\varepsilon = Q - \varepsilon^3 X^3.$$

Therefore: $Q \in \overline{\Omega_P} \setminus P \circ \mathrm{End}(\mathbb{C}^3)$

THE BOUNDARY OF AN ORBIT CLOSURE

NOTATION

- $W \cong \mathbb{C}^n$
- $P \in \mathbb{C}[W]_d \cong \mathbb{C}^N$ where $N = \binom{n+d-1}{d}$
- $\Omega_P := P \circ GL(W)$
- $\partial\Omega_P := \overline{\Omega_P} \setminus \Omega_P$

EXAMPLE

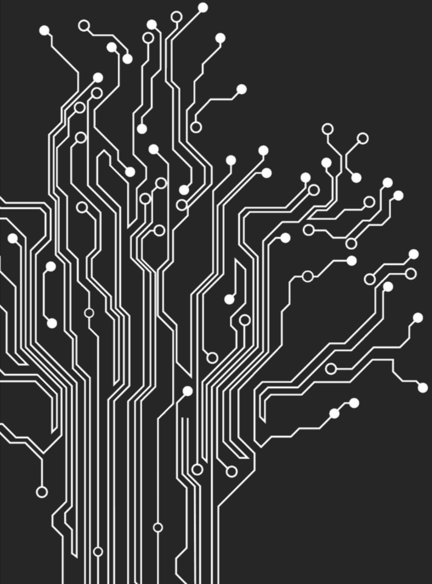
- $W = \mathbb{C}^{d \times d}$
- $P = \det_d$
- $\mathrm{End}(W) = \{a\colon \mathbb{C}^{d\times d} \to \mathbb{C}^{d \times d} \text{ linear}\}$
- $\partial\Omega_P$ not well understood

NOTE

$\partial\Omega_P$ is the interesting part.

THEOREM

$\partial\Omega_{\det_3}$ is a union of two orbit closures.



THE RATIONAL ORBIT MAP

Transitioning from affine cones to projective space

THE POWER OF PROJECTIVE SPACE

PROBLEM

Recall $P \in \mathbb{C}[W]$ and let $\check{\omega}_P : \operatorname{End}(W) \to \overline{\Omega}_P$ be defined by $a \mapsto P \circ a$. We usually view $\overline{\Omega}_P$ through this morphism, but it is not surjective.

PROPOSITION

Let $\gamma : \Gamma \to Y$ be a **projective morphism**, i.e. Γ is a projective variety. Then, $\gamma(Z) \subseteq Y$ is a subvariety for every subvariety $Z \subseteq \Gamma$.

MAIN IDEA

If $\check{\omega}_P$ was projective, it would also be surjective. Let's try to make it projective!

MUSINGS

- Let $P \in \mathbb{C}[W]_d$, and $a \in \mathrm{End}(W)$. For $\lambda \in \mathbb{C}$, we get $\check{\omega}_P(\lambda a) = \lambda^d \cdot \check{\omega}_P(a)$. In other words, $\check{\omega}_P$ maps **lines** to **lines**.
- $\overline{\Omega}_P$ is an affine cone, so $\mathbb{P}\overline{\Omega}_P$ is well-defined.

DEFINITION ATTEMPT

Let $\omega_P : \mathbb{P}\,\mathrm{End}(W) \longrightarrow \mathbb{P}\overline{\Omega}_P, [a] \mapsto [P \circ a]$.

PROBLEM

- Say $P = X_1 X_2 \in \mathbb{C}[X_1, X_2]$ and take $a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then, $P \circ a = 0$.
- The definition $\omega_P([a]) = [P \circ a]$ breaks down: The zero polynomial does not correspond to a point in the projective variety $\mathbb{P}\overline{\Omega}_P$.

RATIONAL MAPS

DEFINITION

Let X and Y be varieties, with X irreducible. A **rational map** from X to Y, denoted $\omega\colon X \dashrightarrow Y$, is a morphism that may be undefined on a closed set.

EXAMPLE

Consider $\omega\colon \mathbb{P}^2 \to \mathbb{P}^2$, mapping $[x : y : z] \mapsto [xy : yz : zx]$. The map is not defined at any of the points $[0 : 0 : 1]$, $[0 : 1 : 0]$, and $[1 : 0 : 0]$.

THE RATIONAL ORBIT MAP

$\omega_P\colon \mathbb{P}\operatorname{End}(W) \dashrightarrow \mathbb{P}\overline{\Omega}_P$, $[a] \mapsto [P \circ a]$. Undefined on the **annihilator** of P:

$$\mathcal{A}_P := \{[a] \in \mathbb{P}\operatorname{End}(W) \mid P \circ a = 0\}$$

ORBIT MAP

Set $\mathrm{E} \coloneqq \mathrm{End}(W)$ and

$$\begin{aligned} \check{\omega}_{\mathrm{P}} : \mathrm{E} &\longrightarrow \overline{\Omega}_{\mathrm{P}} \\ a &\longmapsto \mathrm{P} \circ a \end{aligned}$$

After projectivization:

$$\begin{aligned} \omega_{\mathrm{P}} : \mathbb{P}\mathrm{E} &\dashrightarrow \mathbb{P}\overline{\Omega}_{\mathrm{P}} \\ [a] &\longmapsto [\mathrm{P} \circ a] \end{aligned}$$

Has indeterminacy:

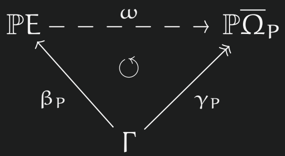
$$\mathcal{A}_{\mathrm{P}} \coloneqq \{[a] \mid \mathrm{P} \circ a = 0\}$$

RESOLVING INDETERMINACY

Define the **graph** of ω_{P}:

$$\Gamma \coloneqq \overline{\{([a], [\mathrm{P} \circ a]) \mid [a] \notin \mathcal{A}_{\mathrm{P}}\}} \subseteq \mathbb{P}\mathrm{E} \times \mathbb{P}\overline{\Omega}_{\mathrm{P}}$$

This **projective variety** has two **morphisms** induced by the projections to each cartesian factor:



The morphism $\gamma_{\mathrm{P}} : \Gamma \longrightarrow \mathbb{P}\overline{\Omega}_{\mathrm{P}}$ is projective.

ORBIT MAP

Set $E \coloneqq \mathrm{End}(W)$ and

$$\begin{aligned} \omega_P : \mathbb{P}E &\dashrightarrow \mathbb{P}\overline{\Omega}_P \\ [a] &\longmapsto [P \circ a] \end{aligned}$$

Resolved by its graph:

$$\begin{array}{ccc} \mathbb{P}E & \overset{\omega}{\dashrightarrow} & \mathbb{P}\overline{\Omega}_P \\ {\scriptstyle \beta_P}\nwarrow & \circlearrowleft & \nearrow{\scriptstyle \gamma_P} \\ & \Gamma & \end{array}$$

BLOWING UP

The morphism $\beta_P : \Gamma \to \mathbb{P}E$ is a so-called blow-up. It is defined by the ideal $I \subseteq \mathbb{C}[E]$ that is generated by the equations $P \circ a = 0$. Note that

$$\mathcal{A}_P = \{[a] \mid P \circ a = 0\} = Z(I)$$

but $I \neq \sqrt{I}$ in general; Γ is not defined $\mathcal{A}_P$, but by the ideal I itself. The ideal I is equivalent to a closed **sub-scheme** $\hat{\mathcal{A}}_P \subseteq \mathbb{P}E$, called the **center** of β_P. If $\hat{\mathcal{A}}_P$ is a variety, then $\hat{\mathcal{A}}_P = \mathcal{A}_P$.

QUICK STRATEGY RECAP

THEOREM

$\partial\Omega_{\det_3}$ is a union of two orbit closures.

PROOF STRATEGY

- Identify two components, both of which are orbit closures.
- Study the map $\omega : \mathbb{P}\,\mathrm{End}(W) \dashrightarrow \mathbb{P}\overline{\Omega}_P$, $[a] \mapsto [P \circ a]$.
- Understand the geometry of $\hat{A}_P$.
- Deduce from this the changes introduced by the blow-up $\beta_P : \Gamma \to \mathbb{P}\,\mathrm{End}(W)$.
- A blow-up introduces new hypersurfaces;
- This yields a bound on the number of components of $\partial\Omega_P$.

BLOW-UP TOY EXAMPLE

Consider $\omega\colon \mathbb{P}^2 \dashrightarrow \left\{ \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \mid t_1 t_4 = t_2 t_3 \right\}$, defined by $\omega([x:y:z]) := \begin{bmatrix} x{\cdot}x & x{\cdot}y \\ x{\cdot}z & y{\cdot}z \end{bmatrix}$. The map is not defined at $a_1 := [0:0:1]$ and $a_2 := [0:1:0]$.

$$L_1 := \left\{ \begin{bmatrix} 0 & 0 \\ s & t \end{bmatrix} \mid [s:t] \in \mathbb{P}^1 \right\} \not\subseteq \mathrm{im}(\omega)$$

Let $p_{s,t}(\varepsilon) := [\varepsilon s : \varepsilon t : 1]$, the line through a_1 in the direction $[s:t] \in \mathbb{P}^1$.

$$\forall [s:t] \in \mathbb{P}^1 : \quad \omega(p_{s,t}(\varepsilon)) = \begin{bmatrix} \varepsilon^2 s^2 & \varepsilon^2 st \\ \varepsilon s & \varepsilon t \end{bmatrix} = \begin{bmatrix} \varepsilon s^2 & \varepsilon st \\ s & t \end{bmatrix} \xrightarrow{\varepsilon \to 0} \begin{bmatrix} 0 & 0 \\ s & t \end{bmatrix}$$

Hence, $L_1 \subseteq \overline{\mathrm{im}(\omega)}$. Equivalently for $L_2 := L_1^T$ by switching y and z.

- The ideal defining this blow-up is $I = (x^2, xy, xz, yz)$.
- However, its **saturation** is $(x, yz) = I(\{a_1, a_2\})$.
- because of this[1], the center of this blow-up is $\{a_1, a_2\}$.

This blow-up replaces each a_i by a copy of $\mathbb{P}^1 \cong L_i$.

[1] Gathmann Lecture Notes 2003, Lemma 5.5.9

THE BLOW-UP PROPOSITION

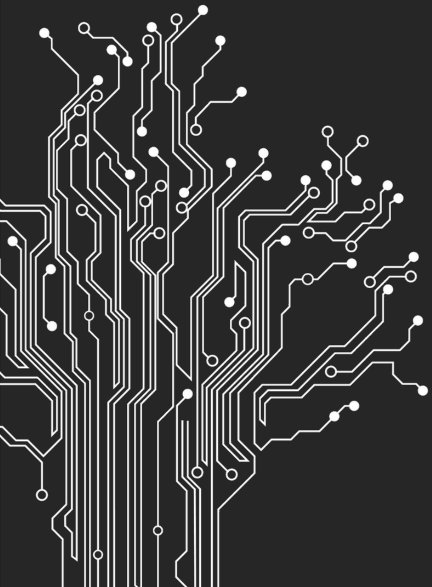
Assume that the blow-up $\beta_P : \Gamma \to \mathbb{P}E$ can be written as a sequence of blowups

$$\begin{array}{ccccccccc} \Gamma = Y_k & \xrightarrow{\beta_k} & Y_{k-1} & \xrightarrow{\beta_{k-1}} & \cdots & \xrightarrow{\beta_2} & Y_1 & \xrightarrow{\beta_1} & \mathbb{P}E \\ & \searrow^{\gamma_P} & & & \overline{\Omega}_P & & & \swarrow^{\omega_P} & \end{array}$$

where the center of each β_i is smooth, then $\partial\Omega_P$ has at most $k+1$ components.

PROOF SKETCH

- A blowup with smooth center creates only one new hypersurface.
- Since γ_P is surjective, k new hypersurfaces were enough to completely cover $\partial\Omega_P$.



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MAXIMAL LINEAR SUBSPACES

Dissecting the indeterminacy of the rational orbit map

MAXIMAL VANISHING LINEAR SUBSPACES

WHAT IS THIS ABOUT

Recall that $P \in \mathbb{C}[W]$. To understand the blow-up $\beta : \Gamma \to \mathbb{P}\,\mathrm{End}(W)$, we have to understand the annihilator of P:

$$\mathcal{A}_P = \{[a] \in \mathbb{P}\,\mathrm{End}(W) \mid P \circ a = 0\} = \{[a] \in \mathbb{P}\,\mathrm{End}(W) \mid \mathrm{im}(a) \subseteq Z(P)\}$$

DEFINITION: MAXIMAL VANISHING LINEAR SUBSPACES

- $\mathrm{End}(W, L) := \{a \in \mathrm{End}(W) \mid \mathrm{im}(a) \subseteq L\}$
- $\overline{\mathcal{L}}_P := \{L \subseteq Z(P) \subseteq W \mid L \text{ linear}\}$
- $\mathcal{L}_P$ the inclusion-wise maximal elements of $\overline{\mathcal{L}}_P$

$$\mathcal{A}_P = \left\{[a] \,\middle|\, \mathrm{im}(a) \in \overline{\mathcal{L}}_P\right\} = \bigcup_{L \in \mathcal{L}_P} \mathbb{P}\,\mathrm{End}(W, L)$$

MAXIMAL VANISHING LINEAR SUBSPACES

EXAMPLE: MAXIMAL VANISHING LINEAR SPACES FOR DET_3

Let $W = \mathbb{C}^{3\times 3}$ and $P = \det_3$. Then, $\mathcal{L}_P = \{L_1, L_2, L_3, L_4\}$ where:

$$L_1 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix} \quad L_4 = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$$

Hence:

$$\mathcal{A}_P = \mathbb{P}\,\mathrm{End}(W, L_1) \cup \mathbb{P}\,\mathrm{End}(W, L_2) \cup \mathbb{P}\,\mathrm{End}(W, L_3) \cup \mathbb{P}\,\mathrm{End}(W, L_4)$$

GOOD NEWS

We will be able to ignore all but L_4 to understand $\det_3$.

BAD NEWS

For $d \geqslant 5$, the set $\mathcal{L}_{\det_d}$ is no longer finite and not entirely understood.

EXPLOITING SEMISTABILITY

Recall the maximal linear subspaces of $Z(\det_3)$:

$$L_1 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix} \qquad L_2 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix} \qquad L_3 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix} \qquad L_4 = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$$

The space L_1 is **unstable**:

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^{-2} \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon \cdot x_1 & \varepsilon \cdot x_2 & \varepsilon \cdot x_3 \\ \varepsilon \cdot x_4 & \varepsilon \cdot x_5 & \varepsilon \cdot x_6 \\ 0 & 0 & 0 \end{pmatrix} = \varepsilon \cdot \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\varepsilon \to 0} 0$$

because $T_\varepsilon = \operatorname{diag}(\varepsilon, \varepsilon, \varepsilon^2)$ satisfies the following conditions:

- $G_P := \{g \in GL(W) \mid P \circ g = P\}$.
- Since $\det(T_\varepsilon) = 1$, we have $\det(T_\varepsilon X) = \det(X)$.
- Define $t_\varepsilon(X) := T_\varepsilon X$, then $t_\varepsilon \in G_P$.

Similar maps exist for L_2 and L_3; but not for L_4.

EXPLOITING SEMISTABILITY

DEFINITION: SEMISTABLE POINTS

$$G_P := \{g \in GL(W) \mid P \circ g = P\}$$ the stabilizer of P

$$\mathcal{N}_P := \{a \in \mathrm{End}(W) \mid 0 \in \overline{G_P a}\}$$ the null cone of G_P acting on $\mathrm{End}(W)$

We then define $E^{ss} := \mathrm{End}(W) \setminus \mathcal{N}_P$ and $\mathbb{P}E^{ss} := \mathbb{P}(E^{ss})$.

APPLICATION

We can replace $\mathbb{P}E$ by $\mathbb{P}E^{ss}$ for the following reasons:

- There is a quotient $\pi : \mathbb{P}E^{ss} \to \mathbb{P}E^{ss} /\!\!/ G_P$ and the variety $\mathbb{P}E^{ss} /\!\!/ G_P$ is projective.
- The rational map $\omega_P : \mathbb{P}E^{ss} \dashrightarrow \mathbb{P}\overline{\Omega}_P$ is G_P-invariant.

INDETERMINACIES OF RATIONAL MAPS

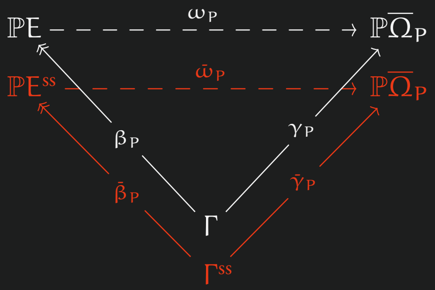
DEFINITION

$$G_P := \{ g \in GL(W) \mid P \circ g = P \}$$
$$\mathcal{N}_P := \{ a \in \mathrm{End}(W) \mid 0 \in \overline{G_p\, a} \}$$
$$E^{ss} := \mathrm{End}(W) \setminus \mathcal{N}_P$$
$$\mathbb{P}E^{ss} := \mathbb{P}(E^{ss})$$

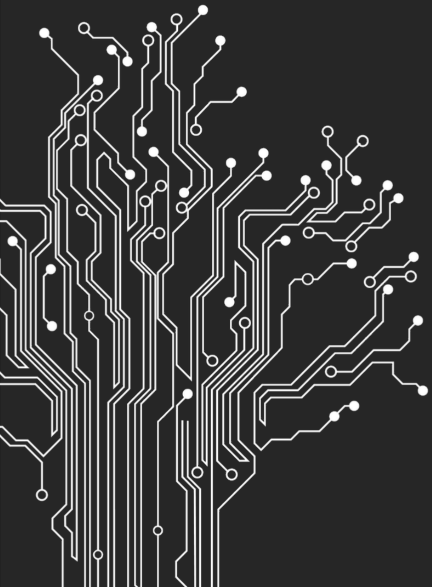
For $P = \det_3$, we have:

$$\mathcal{A}_P^{ss} := \mathcal{A}_P \cap \mathbb{P}E^{ss} \subseteq \mathbb{P}\,\mathrm{End}(W, L_4)$$

ILLUSTRATION



In this case, $\hat{\mathcal{A}}_P^{ss} = \mathcal{A}_P^{ss}$ is a smooth variety, and the center of the blow-up $\bar{\beta}_P$. With the Blow-Up Proposition, this proves that $\partial\Omega_{\det_3}$ has at most two irreducible components.



STRATEGY AND READING SUGGESTIONS

STRATEGY AND READING SUGGESTIONS

READING SUGGESTIONS (MAYBE DATED)

- Gathmann Lecture Notes 2003
- Hanspeter Kraft – Geometrische Methoden in der Invariantentheorie
- Hanspeter Kraft – Geometric methods in representation theory
- Patrice Tauvel & Rupert W. T. Yu – Lie Algebras and Algebraic Groups
- Harm Derksen and Gregor Kemper – Computational Invariant Theory
- My Dissertation
- Don Rosa - The Life and Times of Scrooge McDuck