Discrete Convex Optimization for Left-Right Action (nc-rank & det)

Part I

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Based on joint work with
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Contents

M. Hamada & H. Hirai: Computing the nc-rank via discrete convex optimization on CAT(0) spaces, *SIAM Journal on Applied Geometry and Algebra, 2021.*

H. Hirai & M. Ikeda: A cost-scaling algorithm for computing the degree of determinants, 2020, arXiv.

H. Hirai: Computing the degree of determinants via discrete convex optimization on Euclidean buildings, *SIAM Journal on Applied Geometry and Algebra,* 2019.

0. Edmonds problem, ncrank, left-right action

$\sim$ summarize basic facts
& explain how *submodularity* arises in this context

Edmonds Problem

Edmonds 1967

Can we compute the rank of

$$A = A_1 x_1 + A_2 x_2 + \cdots + A_m x_m$$

in polynomial time ?

x_i : variables, A_i: $n \times n$ matrices over field $\mathbb{K}$

A: matrix over $\mathbb{K}[x_1, x_2, \ldots, x_m] \subset \mathbb{K}(x_1, x_2, \ldots, x_m)$

- RP, but P ? (for large field)
- Related to fundamental problems in diverse areas
  ～ combinatorial optimization, rigidity theory, TCS,...

Non-commutative Edmonds Problem

Ivanyos-Qiao-Subrahmanyam 2017

Can we compute the rank ***(ncrank)*** of

$$A = A_1 x_1 + A_2 x_2 + \cdots + A_m x_m$$

in polynomial time ?

x_i : noncommutative variables, A_i: matrices over field $\mathbb{K}$

A: matrix over free ring $\mathbb{K}\langle x_1, x_2, \dots, x_m \rangle$

$\cap$

free skew field $\mathbb{K}(\langle x_1, x_2, \dots, x_m \rangle)$

- Broaden the literature:
  Noncommutative algebra, Invariant theory, …

Connection to invariant theory

$\mathbb{K} = \mathbb{C}$

Left-right action: $SL_n \times SL_n \ni (S, T) \mapsto (SA_kT)_{k=1,2,\ldots,m}$

- $(A_k)_{k=1,2,\ldots,m}$ belongs to the null cone of this action

$$\Updownarrow$$

- $\mathbf{0} \in$ The closure of the orbit of $(A_k)_{k=1,2,\ldots,m}$ of this action

$$\Updownarrow$$

- $0 =$ The infimum of norm of $(SA_kT)_{k=1,2,\ldots,m}$ over (S, T)

$$\Updownarrow$$

- $\text{ncrank} \sum_k A_k x_k < n$

nc-rank in **P**

- Garg-Gurvits-Oliveira-Wigderson 2019 : $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$

Gurvits' operator scaling $\approx$ alternating minimization for a *geodesically convex* function on symmetric space GL_n/U_n

nc-nonsingularity $\Leftrightarrow$ $\inf \left\{ \log \frac{\det \sum A_i X A_i^T}{\det X} : X \succ 0 \right\} > -\infty$

→ *Noncommutative optimization* (Bürgisser et al.)

- Ivanyos-Qiao-Subrahmanyam 2018: $\mathbb{K}$ arbitrary

Wong sequence --- vector-space analogue of augmenting path

Matrix completion by cyclic division algebra

The contribution of [Hamada-H. 21]

$\sim$ a different polytime algorithm to compute nc-rank

Features / techniques:

- Submodular function optimization on modular lattice
- Geodesically convex optimization on (non-manifold) CAT(0) space

For nc-rank over $\mathbb{Q}$ (to avoid bit-complexity issue):

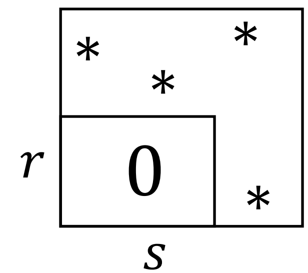
- reduction to nc-rank over $\mathrm{GF}(p)$ by p-adic valuation

  → discrete convex optimization on Euclidean building for $GL_n(\mathbb{Q})$

Submodularity in ncrank

Thm (Fortin-Reutenauer 2004)

$$\text{ncrank} \sum_k A_k x_k = 2n - \text{Max. } r + s$$

s.t. $PA_kQ =$ 

$(\forall k)$

$$P, Q \in \mathrm{GL}_n(\mathbb{K})$$

$$\text{Min. } 2n - \dim X - \dim Y \quad \text{s.t.} \quad A_k(X, Y) = \{0\} \quad (\forall k)$$

$X, Y \subseteq \mathbb{K}^n$ vector subspaces

where $A_k(x, y) := x^T A_k y$

$$Y \leftarrow \cap_k (A_k X)^\perp = (\textstyle\sum_k A_k X)^\perp$$

$\dim(\sum_k A_k X)^\perp = n - \dim \sum_k A_k X$

Min. $n - \dim X + \dim \sum_k A_k X$ s.t. $X \subseteq \mathbb{K}^n$ vector subspaces

Obs. $X \mapsto f(X) := n - \dim X + \dim \sum_k A_k X$ is *submodular*:

$$f(X) + f(X') \geq f(X \cap X') + f(X + X')$$

$\because \dim X + \dim X' = \dim X \cap X' + \dim X + X'$

$$\dim \sum_k A_k X + \dim \sum_k A_k X'$$

$$= \dim(\sum_k A_k X) \cap (\sum_k A_k X') + \dim(\sum_k A_k X) + (\sum_k A_k X')$$

$$\rotatebox{90}{$\subseteq$} \qquad\qquad\qquad\qquad //$$

$$\geq \dim \sum_k A_k (X \cap X') + \dim \sum_k A_k (X + X')$$

1. Background: Submodular function

$\sim$ explain basic facts of (classical) submodular functions
& some results of submodular functions on *lattices*

Submodular function

$[n] := \{1,2,\ldots,n\}, \quad \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$

Def: $f: 2^{[n]} \to \overline{\mathbb{R}}$ is submodular $\Leftrightarrow$

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \quad (X, Y \subseteq [n])$$

Usually assume $f(\emptyset) = 0$

- Fundamental importance in combinatorial optimization from 60' (Edmonds)
- Generalize matroid (rank function) & greedy algorithm

Max. $w^T x$

s.t. $x(X) \leq f(X) \; (\forall X)$

$x(V) = f(V)$

$x \in \mathbb{R}^n$

$x(X) := \sum_{i \in X} x_i$

Linear optimization over "base polytope" of f

solved by: $w_1 \geq w_2 \geq \cdots \geq w_n$

$$x_i := f([i]) - f([i-1])$$

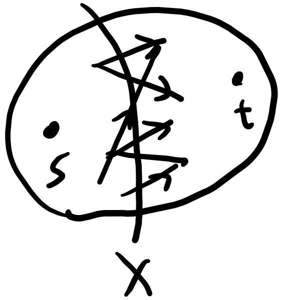
This generalizes matroid greedy algorithm

Min-max theorems in combinatorial optimization are often written by submodular functions

Max-flow min-cut theorem (Ford-Fulkerson)

Max. flow-value of f $=$ $\text{Min } c(\delta X) := \sum_{e\in\delta X} c(e)$

s.t. $s \in X \not\ni t$

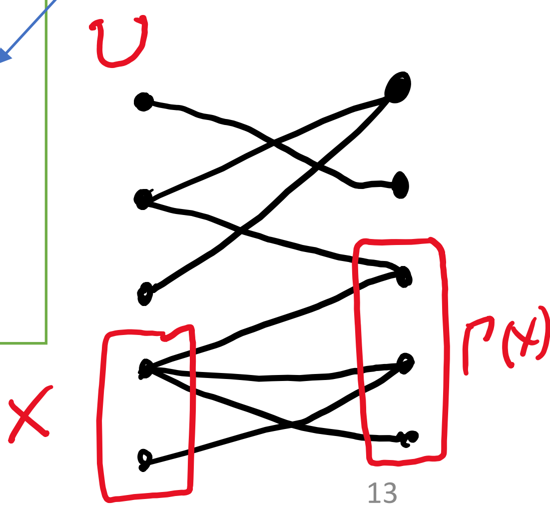


submodular

Bipartite matching (Hall, Konig-Egervary)

Max. $|M|$ $=$ $\text{Min. } |U| - |X| + \Gamma(X)$

s.t. M:matching　　　　s.t. $X \subseteq U$



Submodular Function Minimization (SFM)

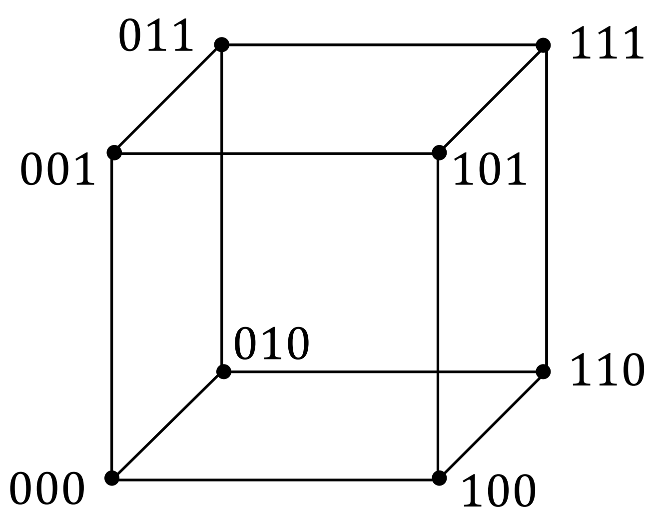
Min. $f(X)$ s.t. $X \subseteq [n]$

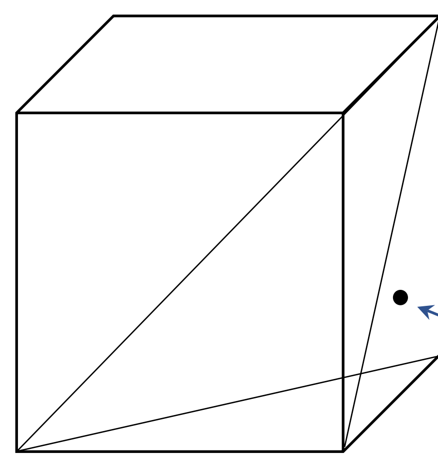
$f: 2^{[n]} \to \overline{\mathbb{R}}$: $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$

$X \leftrightarrow \mathbf{1}_X \coloneqq \sum_{i \in X} e_i$

$f: \{0,1\}^n \to \overline{\mathbb{R}}$: $f(x) + f(y) \geq f(\min(x,y)) + f(\max(x,y))$

$\bar{f}: [0,1]^n \to \overline{\mathbb{R}}$ *Lovász extension*, convex





$\bar{f}(x) = \lambda_0 f(000) + \lambda_1 f(100)$
$+ \lambda_2 f(110) + \lambda_3 f(111)$

$x = \lambda_0(000) + \lambda_1(100)$
$+ \lambda_2(110) + \lambda_3(111)$

Min f over $\{0,1\}^n$ $\equiv$ Min $\bar{f}$ over $[0,1]^n$

Polynomial time algorithm $\Leftarrow$ Ellipsoid method

Subgradient of $\bar{f}$
$\Uparrow$
greedy algorithm

Grotschel-Lovász-Schrijver 81

Lovász extension ~ submodular function and convexity

$f:\{0,1\}^n \to \overline{\mathbb{R}}$

Def. Lovász extension $\bar{f}:[0,1]^n \to \overline{\mathbb{R}}$

$$\bar{f}(x) \coloneqq \sum_i \lambda_i f\left(\mathbf{1}_{X_i}\right) \quad (x \in [0,1]^n)$$

where $x = \sum_i \lambda_i \mathbf{1}_{X_i}$

$$X_1 \subset X_2 \subset \cdots \subset X_n = [n]$$

$$(0.8\ \ 0.2\ \ 0.5\ \ 0.3) = 0.2\ (0100) + 0.1\ (0101) + 0.2\ (0111) + 0.3\ (1111)$$

$$\bar{f}(0.8\ \ 0.2\ \ 0.5\ \ 0.3) = 0.2 f(0100) + 0.1 f(0101) + 0.2 f(0111) + 0.3 f(1111)$$

Thm. [Lovász]

$f:\{0,1\}^n \to \overline{\mathbb{R}}$ is submodular $\Leftrightarrow$ $\bar{f}:[0,1]^n \to \overline{\mathbb{R}}$ is convex

Variants of submodular functions extensible to convex functions

Bisubmodular function $f: \{-1,0,1\}^n \to \mathbb{R}$

$$f(x) + f(y) \geq f(x \sqcap y) + f(x \sqcup y)$$

$\exists$ convex extension on $[-1,1]^n$, minimizable in polytime

L-convex function $f: \mathbb{Z}^n \to \mathbb{R}$

$$f(x) + f(y) \geq f(\min(x,y)) + f(\max(x,y))$$

$$f(x + \mathbf{1}) = f(x)$$

$\exists$ convex extension on $\mathbb{R}^n$, successive descent by SFM

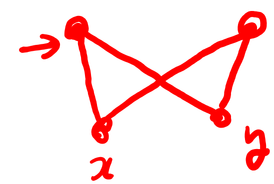
→ Discrete Convex Analysis (Murota 90'～)

Submodular functions on lattices

Def. Lattice := partially ordered set (poset) $(\mathcal{L}, \preccurlyeq)$ s.t.

join $x \vee y = \min \{z \in \mathcal{L} \mid x \preccurlyeq z \succcurlyeq y\}$ &
meet $x \wedge y = \max \{z \in \mathcal{L} \mid x \succcurlyeq z \preccurlyeq y\}$ exist

Ex. $\left(2^{[n]}, \subseteq\right)$, $X \wedge Y = X \cap Y$, $X \vee Y = X \cup Y$



Ex. $(\mathcal{S}(\mathbb{K}^n), \subseteq)$, $X \wedge Y = X \cap Y$, $X \vee Y = X + Y$

family of all vector subspaces of $\mathbb{K}^n$

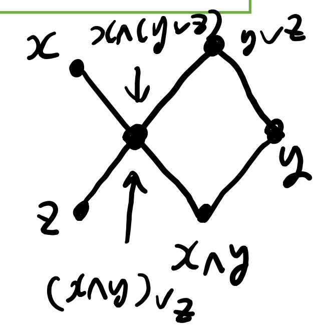
Def. Submodular function on lattice $\mathcal{L}$

$$\Leftrightarrow \quad f: \mathcal{L} \to \overline{\mathbb{R}}, \; f(x) + f(y) \geq f(x \wedge y) + f(x \vee y)$$

In this view, the original submodular function $f: 2^{[n]} \to \overline{\mathbb{R}}$
is submodular function on Boolean / distributive lattice

Submodular functions on modular lattices

Modular lattice := lattice $\mathcal{L}$ s.t. $x \succcurlyeq z \Rightarrow (x \wedge y) \vee z = x \wedge (y \vee z)$



Ex. $\mathcal{S}(\mathbb{K}^n)$ lattice of all vector subspaces of $\mathbb{K}^n$

ncrank $\sum_k A_k x_k$ = Min. $n - \dim X + \dim \sum_k A_k X$

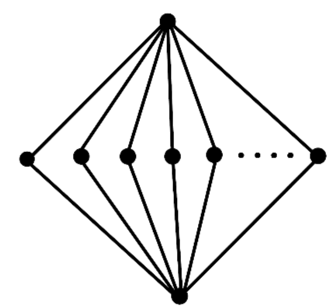
s.t. $X \subseteq \mathbb{K}^n$ vector subspaces

is SFM on modular lattice $\mathcal{S}(\mathbb{K}^n)$

Currently, little is known on tractability of SFM on modular lattices.
*So, **ncrank** is interesting from submodular function theory.*

One positive result [Fujishige et al. 14]

SFM on $\mathcal{L}^n$ for finite rank-2 modular lattice $\mathcal{L}$ is solvable in poly$(n, |\mathcal{L}|)$- time



- For SFM on modular lattice,
  nice convex relaxation on $\mathbb{R}^n$, i.e., Lovász extension, is not known.

- Recent study [H. 18] showed
  SFM on modular lattice admits *convex relaxation on a CAT(0)-space.*

- [Hamada-H 21] applied this relaxation to SFM of ncrank.

2. Background: CAT(0) space / Hadamard space

- definition & examples of CAT(0) spaces
- CAT(0) complexes associated with lattices
- convex relaxation of SFM on modular lattice

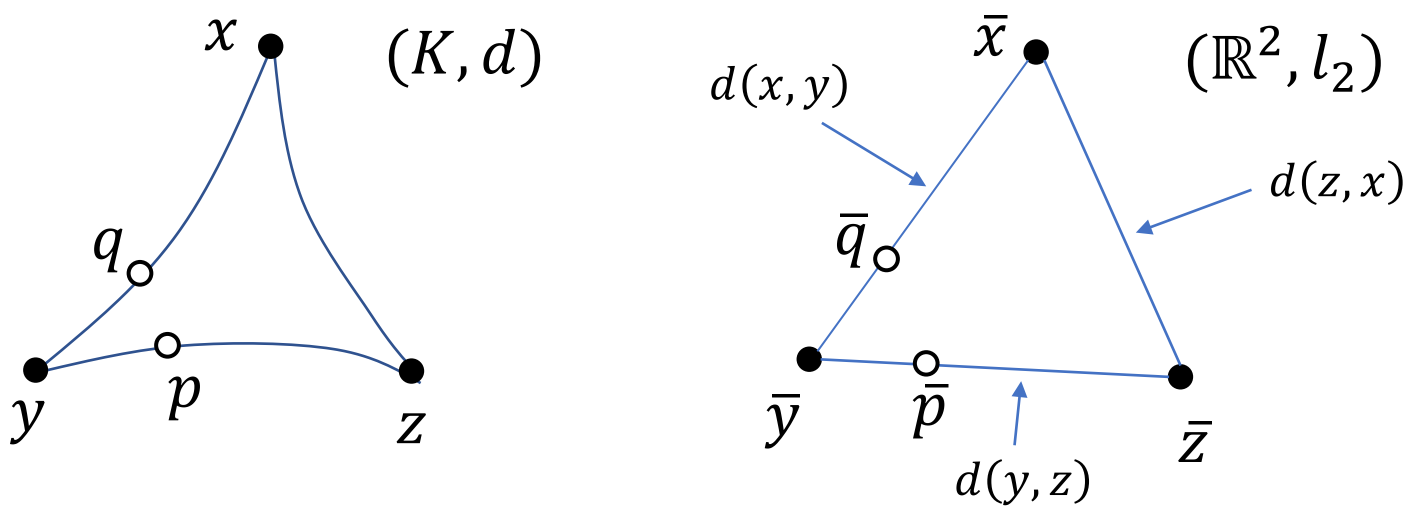
(K, d): metric space

Geodesic := path γ: $[0,1] \to K$ with $d\big(\gamma(s), \gamma(t)\big) = |s - t| d(\gamma(0), \gamma(1))$

(K, d): geodesic $\Leftrightarrow$ any of two points can be joined by a geodesic

CAT(0) space (Gromov 1987)

A geodesic metric space (K, d) s.t. every triangle is "slimmer"



CAT(0)-inequality: $d(p, q) \leq \|\bar{p} - \bar{q}\|_2$

Def: Hadamard space := complete CAT(0) space

Textbook:
B.R. Bridson & A.Haefliger: *Metric Spaces of Non-Positive Curvature,* Springer, 1999.

Examples

$\mathbb{R}^n$, hyperbolic space,…

Symmetric space of nonpositive curvature / noncompact type

$\mathbb{R}^k \times G/K$ (G: semisimple Lie group, K: maximal compact)

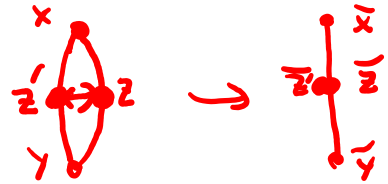
Ex. $GL_n(\mathbb{C})/U_n = P_n$

Hadamard manifold = simply-connected complete Riemannian manifold of nonpositive sectional curvature

Tree, Euclidean building,…

Ex. $GL_n(\mathbb{K}(t))/GL_n(\mathbb{K}(t)^-)$

Thm. A CAT(0) space is uniquely geodesic, i.e.,
any of two points can be joined by a unique geodesic.



Def. $f : K \to \mathbb{R}$ is (geodesically) convex $\Leftrightarrow$

$$(1-t)f(x) + tf(y) \geq f(\gamma(t)) \quad (x, y \in K, t \in [0,1])$$

γ: unique geodesic connecting x, y

Convex optimization theory on CAT(0) space / Hadamard space is now developing but still undeveloped.

M. Bačák: *Convex Analysis and Optimization in Hadamard Spaces,* De Gruyter, 2014

Modeling & Algorithm & Optimization on CAT(0) spaces

- The space of phylogenetic trees (BHV-tree space) [Billera et al 2001]
- Configuration spaces of robots [Abram &Ghrist 2004]
- Polynomial-time geodesic computation

  for BHV-tree space [Owen 2011]

  for CAT(0) cubical complex [Hayashi 2021]
- Proximal point algorithm for convex optimization [Bačák 2014]
- Combinatorially-defined CAT(0) spaces:

  median graph → CAT(0) cubical complex [Chepoi 2000]

  poset / lattice → orthoscheme complex [Brady-McCammond 2012]

Orthoscheme Complex Brady-McCammond 2012

$\mathcal{L}$: graded poset of finite rank

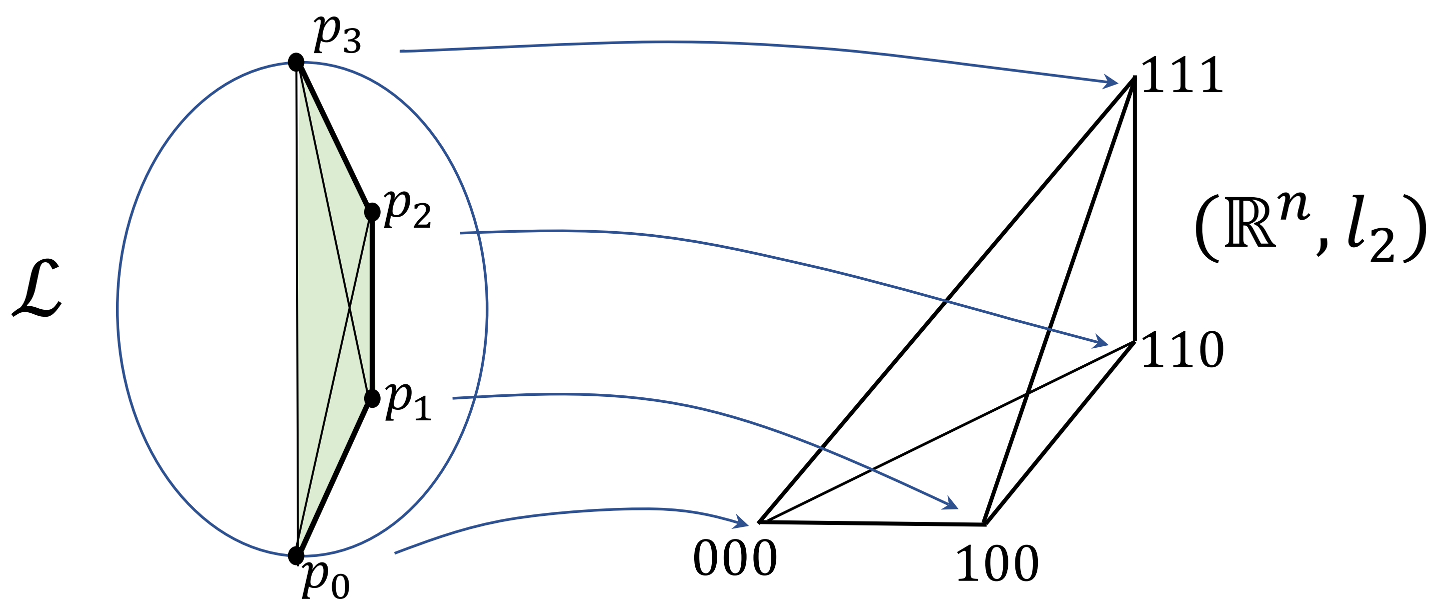
formal convex combination $\sum_i \mu_i = 1, \mu_i \geq 0$

$$K(\mathcal{L}) := \{ \ \sum_i \mu_i p_i \mid p_0 \prec p_1 \prec \cdots \prec p_n, \ \ p_i \in \mathcal{L} \}$$

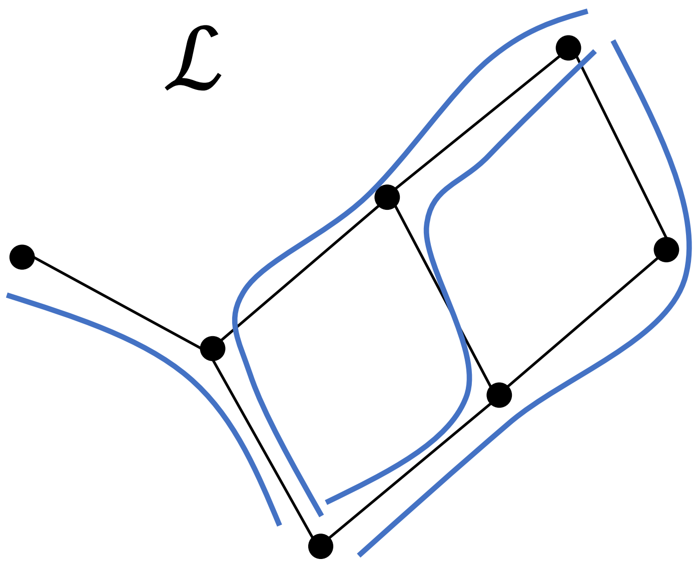
= the order complex of $\mathcal{L}$

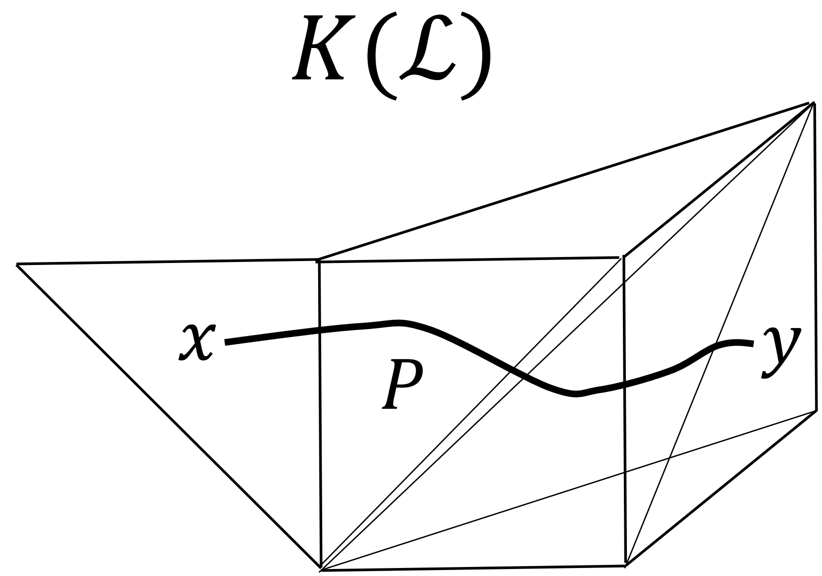
metrized by filling *orthoscheme* to each simplex:

$$\sum_i \mu_i p_i \longmapsto \sum_i \mu_i (\overbrace{1, \dots, 1}^{i}, 0, \dots, 0)$$



Orthoscheme = conv $\{(0,0,\dots,0), (1,0,\dots,0), (1,1,0,\dots,0), \dots, (1,1,1,\dots,1)\}$





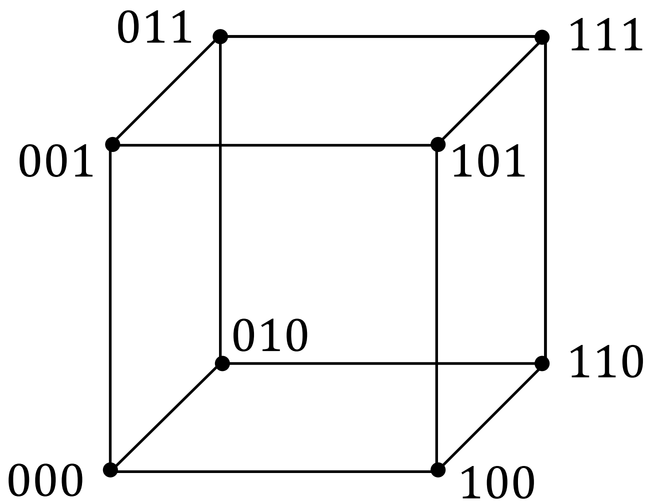
The length $d(P)$ of a path $P: [0,1] \to K(\mathcal{L})$ is well-defined

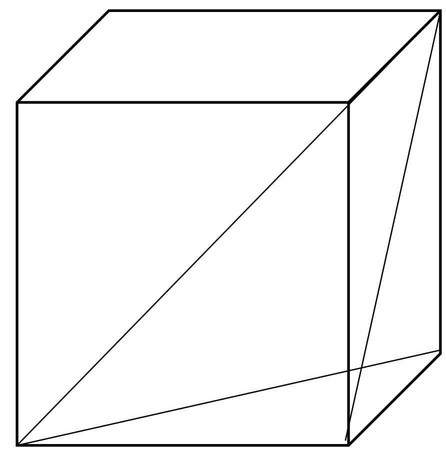
$$d(x, y) \coloneqq \inf\{d(P) \mid P: (x, y)\text{−path}\}$$

$\to (K(\mathcal{L}), d)$ is a complete geodesic metric space

Question [Brady-McCammond 2012]:
Which is the class of posets $\mathcal{L}$ having CAT(0) $K(\mathcal{L})$?

Boolean Lattice





$\mathcal{L} = \{0,1\}^n = 2^{[n]}$

$K(\mathcal{L}) \simeq [0,1]^n$

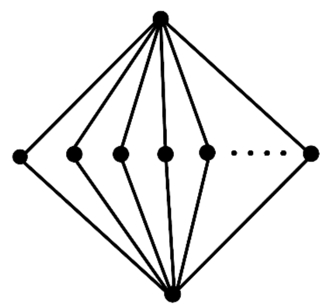
$\rightarrow$ CAT(0)

$\mathcal{L}$: distributive lattice → $K(\mathcal{L})$: order polytope, CAT(0)

Thm [Haettel-Kielak-Schwer 2015, Chalopin-Chepoi-H-Osajda 2020]

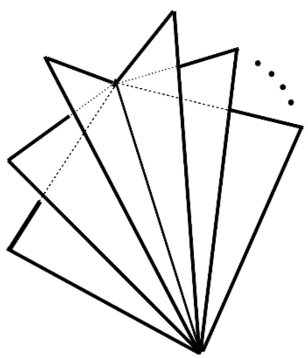
$\mathcal{L}$: modular lattice of finite rank

$\Rightarrow K(\mathcal{L})$ is CAT(0) (→ Hadamard space)



$\mathcal{L}$: modular lattice of subspaces of $\mathbb{K}^2$

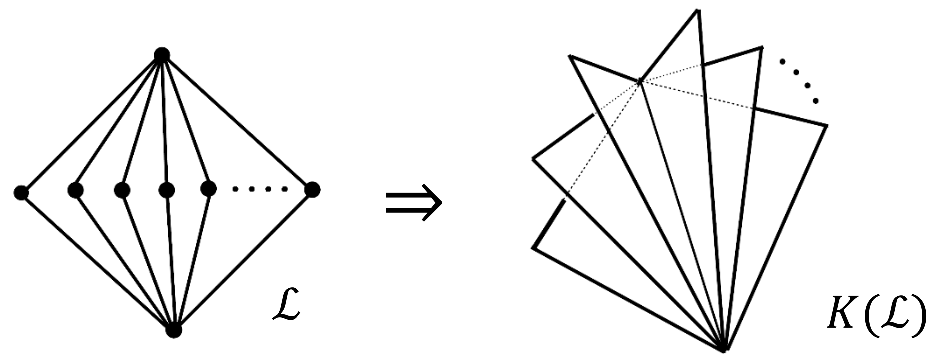
$\Rightarrow$



$K(\mathcal{L})$

→ *We can consider convexity on* $K(\mathcal{L})$

Submodularity and convexity



$f : \mathcal{L} \to \mathbb{R}$

Def. Lovász extension $\bar{f} : K(\mathcal{L}) \to \mathbb{R}$

$$\bar{f}(x) := \sum_i \mu_i f(p_i) \quad \left(x = \sum_i \mu_i p_i \,;\, p_0 \prec p_1 \prec \cdots \prec p_n \right)$$

Thm [H.18] $\mathcal{L}$: modular lattice

$f : \mathcal{L} \to \mathbb{R}$ is submodular $\Leftrightarrow$ $\bar{f} : K(\mathcal{L}) \to \mathbb{R}$ is convex

Classical Lovász extension: $f : \{0,1\}^n \to \mathbb{R}$ is submodular $\Leftrightarrow$ $\bar{f} : [0,1]^n \to \mathbb{R}$ is convex

Lattice theory behind this

The product of lattices of vector subspaces

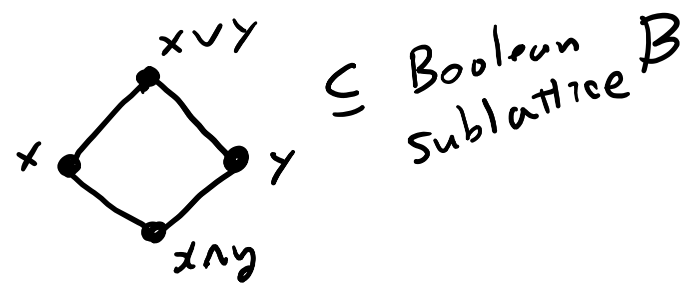
$\wr$

Lem. (Birkhoff-Dedekind) $\mathcal{L}$: "complemented" modular lattice of rank n

- For two chains C, D, there is a Boolean sublattice $\mathcal{B} \supseteq C, D$.
- There is a rank & order-preserving retraction $\mathcal{L} \to \mathcal{B}$

- nonexpansive retraction $K(\mathcal{L}) \to K(\mathcal{B}) \simeq [0,1]^n$
- $K(\mathcal{B})$ is a convex subspace of $K(\mathcal{L})$

$f: \mathcal{L} \to \mathbb{R}$ is submodular
$\Leftrightarrow f: \mathcal{B} \to \mathbb{R}$ is submodular for $\forall \mathcal{B}$
$\Leftrightarrow \bar{f}: K(\mathcal{B}) \to \mathbb{R}$ is convex for $\forall \mathcal{B}$
$\Leftrightarrow \bar{f}: K(\mathcal{L}) \to \mathbb{R}$ is convex



Rem. The lemma is part of the axiom of *building*

Thm (Birkhoff, Tits) complemented modular lattice
$\equiv$ spherical building of type A

Nc-rank is convex optimization on CAT(0) space

$$\text{ncrank} \sum_k A_k x_k = \text{Min.} \quad n - \dim X + \dim \sum_k A_k X$$
$$\text{s.t.} \quad X \in \mathcal{S}(\mathbb{K}^n)$$

$\Downarrow$ Lovász extension

$$\text{ncrank} \sum_k A_k x_k = \text{Min.} \quad n - \overline{\dim}(x) + \overline{\dim \sum_k A_k}\,(x)$$
$$\text{s.t.} \quad x \in K(\mathcal{S}(\mathbb{K}^n))$$

→ Convex optimization on CAT(0) space (Hadamard space)

But, convex optimization on CAT(0) space is undeveloped.
We have no powerful method such as ellipsoid method.

3. Discrete convex optimization for nc-rank

- Splitting Proximal Point Algorithm (SPPA)
- Its convergence
- Application to ncrank
- Application to other problems

Terminology for convex optimization

(K, d): Hadamard space, $f: K \to \mathbb{R}$

L-Lipschitz: $|f(x) - f(y)| \leq L\, d(x, y)$

ε-strongly convex for $\varepsilon > 0$:

$$(1-t)f(x) + tf(y) \geq f\big(\gamma(t)\big) + \frac{\varepsilon}{2}\, t(1-t)d(x,y)^2$$

Lem [see Bačák's book]

- $\exists$ unique minimizer for a strongly convex function
- $x \to d(x, z)^2$ is 2-strongly convex & $2\mathrm{diam}\, K$-Lipschitz

$\wr\wr$

CAT(0)-inequality

Splitting Proximal Point Algorithm Bačák 2014

(K,d): Hadamard space, $f_1, f_2, \ldots, f_N$: convex functions on K

Goal: Minimize $f = \sum_{i=1}^{N} f_i$

SPPA: $x_{k+1} \longleftarrow \operatorname{argmin}_{z\in K} \; f_{k \bmod N}(z) + \frac{1}{2\lambda_k} d(z, x_k)^2$

Thm (Ohta-Pálfia 2015)

f_i: L-Lipschitz, f: ε-strongly convex, $\lambda_k := \frac{1}{\varepsilon(k+1)}$

$\Rightarrow d(x_{kN}, x^*)^2 \le \mathrm{poly}(L, N, \varepsilon^{-1}) \frac{\log k}{k}$

→ $f(x_{kN}) - \min f \le \mathrm{poly}(L, N, \varepsilon^{-1}) \sqrt{\frac{\log k}{k}}$

We do not know whether SPPA is applicable to

$$\operatorname{ncrank} \sum_k A_k x_k = n + \begin{array}{l} \text{Min.} -\overline{\dim}(x) + \overline{\dim \sum_k A_k}(x) \\ \text{s.t.} \quad x \in K(\mathcal{S}(\mathbb{K}^n)) \end{array}$$

Instead, we consider Lovász extension of

$$\begin{array}{ll} \text{Min.} \ -\dim X - \dim Y & \text{s.t.} \quad A_k(X,Y) = \{0\} \quad (\forall k) \\ & \qquad X, Y \in \mathcal{S}(\mathbb{K}^n) \end{array}$$

|||

$$\begin{array}{l} \text{Min.} \ -\dim X - \dim Y + (2n+1) \sum_k \operatorname{rank} A_k|_{X \times Y} \\ \qquad \text{s.t.} \ (X,Y) \in \mathcal{L} \times \mathcal{L}, \ \mathcal{L} := \mathcal{S}(\mathbb{K}^n) \end{array}$$

partial order reversed

Lem: The objective function is submodular in $\mathcal{L} \times \mathcal{L}^*$

$$f(X,Y) + f(X',Y') \geq f(X+X', Y \cap Y') + f(X \cap X', Y + Y')$$

$\wedge = (+, \cap), \ \vee = (\cap, +)$

$$\text{Min.} \quad -\dim X - \dim Y + (2n+1) \sum_i \operatorname{rank} A_i|_{X \times Y}$$

$$\text{s.t.} \quad (X,Y) \in \mathcal{L} \times \mathcal{L}^*$$

$\Downarrow$ Lovász extension

$$\text{Min.} \quad -\overline{\dim}(x) - \overline{\dim}(y) + (2n+1) \sum_i \overline{\operatorname{rank} A_i}(x,y)$$

$$\text{s.t.} \quad (x,y) \in K(\mathcal{L} \times \mathcal{L}^*) = K(\mathcal{L}) \times K(\mathcal{L}^*)$$

Perturbation term

$$+\frac{1}{4n}\left(d(0,x)^2+d(0,y)^2\right)$$

$$\text{Min.} \quad -\overline{\dim}(x)-\overline{\dim}\,(y)+(2n+1)\textstyle\sum_i \overline{\operatorname{rank} A_i}(x,y)$$

$$\text{s.t.} \quad (x,y)\in K(\mathcal{L}\times\mathcal{L}^*)=K(\mathcal{L})\times K(\mathcal{L}^*)$$

- Apply SPPA with:

$$f_0(x,y) := -\overline{\dim}(x)-\overline{\dim}(y)+\frac{1}{4n}\left(d(0,x)^2+d(0,y)^2\right)$$

$$f_i(x,y) := (2n+1)\,\overline{\operatorname{rank} A_i}(x,y) \quad (i=1,2,\ldots,m)$$

$\operatorname{diam} K(\mathcal{L})=\sqrt{n}$

$L=\operatorname{poly}(n,m)$

$\varepsilon=\dfrac{1}{2n}$

- After $k=\operatorname{poly}(n,m)$ iterations, we have $\tilde{f}(x_k,y_k)-\min\tilde{f} \; < 1/2$

$$\rightarrow \bar{f}(x_k,y_k)-\min f<1$$

$$f(x_k,y_k)-\min f \le \tilde{f}(x_k,y_k)-\min\tilde{f}+\min\tilde{f}-\tilde{f}(x^*,y^*)+\frac{1}{4n}\left(d(0,x^*)^2+d(0,y^*)^2\right)<1$$

- By integrality of f, an optimum (X^*,Y^*) exists

in the support of $(x_k,y_k)=\sum_i \lambda_i(X_i,Y_i)$

Main technical contribution of [Hamada-H. 21]

Thm: SPPA is implementable in polytime, i.e.,

$$\text{Min. } f_i(x,y) \;+ \frac{1}{2\lambda}\left(d(x,x_0)^2 + d(y,y_0)^2\right)$$

$$\text{s.t. } (x,y) \in K(\mathcal{L} \times \mathcal{L}^*) = K(\mathcal{L}) \times K(\mathcal{L}^*)$$

where $f_i(x,y) = -\overline{\dim}(x) - \overline{\dim}(y) + C\left(d(0,x)^2 + d(0,y)^2\right)$

or $f_i(x,y) = C\,\overline{\operatorname{rank} A_i}(x,y)$

is solvable in polytime.

Proof: Lattice theoretic / building theoretic argument

- An optimum exists in an *apartment* = the subcomplex of a Boolean sublattice including the supports of $x_0, y_0, x_0^{\perp_{A_i}}, y_0^{\perp_{A_i}}$
- In the apartment $\approx [0,1]^{2n}$, the problem is an easy convex quadratic program

Good / bad points

- Conceptually simple, described uniformly in an arbitrary field
- Applications to other problems ?

- Polytime, but very slow:

  ← Less use of characteristics of the objective function

  ← Undevelopment of optimization theory in CAT(0) spaces

- Bit complexity problem for $\mathbb{K} = \mathbb{Q}$:

  ← Polynomial number of vector-space operations $X \cap X', X + X'$

  can cause exponential bit explosion of bases

Application to related problems

(p,q)-scalability of $\{A_k\}$ by triangular matrices [Franks 2018]

$$\text{Max.} \quad \textstyle\sum_i (p_i - p_{i+1}) \dim E_i \cap X + \sum_i (q_i - q_{i+1}) \dim F_i \cap Y$$

E_i, F_i:standard flags

$$\text{s.t.} \quad A_k(X,Y) = \{0\} \quad (\forall k)$$

X, Y: vector subspaces

Brascamp-Lieb separation for BL-data (B,p) [Garg et al. 2018]

$$\text{Max.} \quad \dim X + \textstyle\sum_i p_i \dim Y_i$$

$$\text{s.t.} \quad B_i(X, Y_i) = \{0\} \quad (\forall i)$$

X, Y_i: vector subspaces

The objectives are sum of supermodular functions.

SPPA approach is applicable, but it only gives pseudo-polynomial complexity

G: connected reductive algebraic group over $\mathbb{C}$,
$K \subseteq G$: maximal compact

$\pi: G \to GL(N, \mathbb{C})$: rational representation

Null-cone membership: $0 \overset{?}{\notin} \overline{\pi(G)v}$

$\Updownarrow$

Symmetric space
of nonpositive curvature

$$-\infty \overset{?}{<} \inf \log \|\pi(g)v\| \text{ s.t. } gK \in G/K$$

geodesically convex

I found:

Unboundedness ← geodesically convex optimization on another CAT(0) space obtained from <u>the *boundary of G/K at infinity*</u>

Ex: Boundary of $GL_n/U_n \approx \{\sum_i \alpha_i X_i \mid X_1 \subset X_2 \subset \cdots \subset X_n = \mathbb{C}^n, \alpha_i \geq 0\} \fallingdotseq K(\mathcal{L})$

I am now writing a note explaining this fact.
Still, I don't know whether SPPA approach is applicable.

Additional references

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