Kronecker and plethysm coefficients in GCT

Greta Panova

University of Southern California

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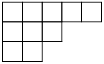
Outline

- Basics from Representation theory
- Kronecker coefficients of the Symmetric group
- Permanent vs determinant via GCT: obstructions
- Positivity of Kronecker coefficients
- Kronecker vs Plethysm
- Beyond positivity

Combinatorics and Reprentation Theory basics

Symmetric group S_n: Permutations $\pi : [1..n] \rightarrowtail [1..n]$ under composition.
Integer partitions $\lambda \vdash n$:

$$\lambda = (\lambda_1, \ldots, \lambda_\ell),\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0,\ \lambda_1 + \lambda_2 + \cdots = n$$

Young diagram of λ:  Here $\lambda = (5, 3, 2)$

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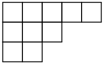
Young diagram of λ:  Here $\lambda = (5, 3, 2)$

Representations of S_n: group homomorphisms $S_n \to GL(V)$,

Example: if $V = \mathbb{C}^3$, $\pi \in S_3$, set $\pi(e_i) := e_{\pi_i}$ for $i = 1..3$, so e.g. $231 \to \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

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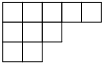
Young diagram of λ:  Here $\lambda = (5, 3, 2)$

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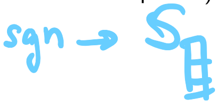
Irreducible decomposition: minimal S_n-invariant subspaces V_i, so
$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$, e.g. $V = \underbrace{\mathbb{C}\langle e_1 + e_2 + e_3 \rangle}_{V_1} \oplus \underbrace{\mathbb{C}\langle e_1 - e_2, e_2 - e_3 \rangle}_{V_2}$

The **irreducible modules (representations)** (up to equivariant isomorphisms) of S_n are the **Specht modules** $\mathbb{S}_\lambda$, indexed by all $\lambda \vdash n$
e.g. $V_1 \simeq \mathbb{S}_{(3)}$ and $V_2 \simeq \mathbb{S}_{(2,1)}$



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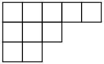
Young diagram of λ:  Here $\lambda = (5, 3, 2)$

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Basis for $\mathbb{S}_\lambda$: Standard **Y**oung **T**ableaux of shape λ: $\lambda = (3, 2)$

$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}$ $\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}$ $\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}$ $\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}$ $\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$

$f^\lambda = \dim \mathbb{S}_\lambda = \frac{n!}{\prod h_u}$

Characters of S_n irreps

Irreducible representation $\rho^\lambda : S_n \to GL(V), V \simeq \mathbb{S}_\lambda$

Example: $V_2 = \mathbb{C}\langle \underbrace{e_1 - e_2}_{v_1}, \underbrace{e_2 - e_3}_{v_2} \rangle \simeq \mathbb{S}_{(2,1)}$:

$\pi \in S_n$

$$\rho^{(2,1)}(231) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$Tr(\rho^{(2,1)})(231) = -1 = \chi^\lambda(\pi)$

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Theorem (Murnaghan-Nakayama rule)

We have that

$$\chi^\lambda[\alpha] = \sum_{T \in MN(\lambda;\alpha)} (-1)^{ht(T)},$$

where $MN(\lambda; \alpha)$ is the set of border-strip tableaux of shape λ, type α with weakly increasing entries along rows and columns; and $ht(T)$ is the total height of the border strips minus $\ell(\alpha)$.

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Example: $\chi^{(2,1)}[(3)] = \underset{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}}{-1}$, $\chi^{(2,1)}[(2,1)] = \underset{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}}{1} + \underset{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array}}{-1} = 0$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 3 & 3 & \\ \cline{1-4} 2 & 2 & & & \\ \cline{1-2}\end{array} \in MN((5,3,2);(4,3,4))$$

Schur functions

Irreducible (polynomial) representations of the **General Linear group** $GL_N(\mathbb{C})$:

Weyl modules V_λ (aka $\mathcal{W}_\lambda$), indexed by highest weights λ, $\ell(\lambda) \leq N$.

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Weyl's determinantal formula:

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det \left[x_i^{\lambda_j + N - j} \right]_{ij=1}^{N}}{\prod_{i<j}(x_i - x_j)}$$

Semi-Standard Young tableaux of shape λ :

$$s_{(2,2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2.$$

$x_1^2 x_2^2$	$x_1^2 x_3^2$	$x_2^2 x_3^2$	$x_1^2 x_2 x_3$	$x_1 x_2^2 x_3$	$x_1 x_2 x_3^2$
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1 1 / 2 2	1 1 / 3 3	2 2 / 3 3	1 1 / 2 3	1 2 / 2 3	1 2 / 3 3

Schur-Weyl duality: $GL(V) \times S_n$ action

$$V^{\otimes k} = \oplus_\lambda V_\lambda \otimes \mathbb{S}_\lambda.$$

Frobenius characteristic map: $ch(\chi^\lambda) = s_\lambda$.

Multiplicities

Von Neumann et al, ca. 1934, representations of Lie groups:

$GL_N(\mathbb{C})$ acts on V_λ, V_μ and their tensor product:

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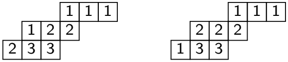

(LR tableaux of shape $(6,4,3)/(3,1)$ and type $(4,3,2)$. $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

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Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

E.g.: $\mathbb{S}_{(2,1)} \otimes \mathbb{S}_{(2,1)} = \mathbb{S}_{(3)} \oplus \mathbb{S}_{(2,1)} \oplus \mathbb{S}_{(1,1,1)}$ and so $g((2,1),(2,1),\nu) = 1$ for $\nu = (3), (2,1), (1,1,1)$.

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In terms of $GL(\mathbb{C}^m)$ modules V_λ, V_μ, V_ν (Schur-Weyl duality):

$$\mathrm{Sym}(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m) = \oplus_{\lambda,\mu,\nu} g(\lambda,\mu,\nu) V_\lambda \otimes V_\mu \otimes V_\nu$$

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Plethysm coefficients in GL-representation compositions:

$$GL_N \to GL(V_\mu) \to GL(V_\nu) \quad \Longleftrightarrow \quad GL_N \to V_\nu[V_\mu] = \oplus_\lambda V_\lambda^{\oplus a_\lambda(\nu[\mu])}$$

State of the art: computation

Littlewood-Richardson:

LR: Input: λ, μ, ν Output: $c^{\lambda}_{\mu\nu}$

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KRON: Input: λ, μ, ν Output: $g(\lambda, \mu, \nu)$
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Kron: Input: λ, μ, ν Output: $g(\lambda, \mu, \nu)$
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[Bürgisser-Ikenmeyer, Pak-P]: $\text{Kron} \in \texttt{GapP}$.
[Ikenmeyer-Mulmuley-Walter]: KronPos is NP-hard.

State of the art: combinatorics

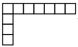
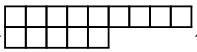
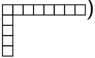
Problem (Murnaghan, 1938, then Stanley et al)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$. Alternatively, show that KRON *is in* $\#$P .

Theorem [Murnaghan] If $|\lambda| + |\mu| = |\nu|$ and $n > |\nu|$, then

$$g((n + |\mu|, \lambda), (n + |\lambda|, \mu), (n, \nu)) = c^{\nu}_{\lambda\mu}.$$

Combinatorial formulas for $g(\lambda, \mu, \nu)$, when:

- μ and ν are hooks (), [*Remmel, 1989*]
- $\nu = (n - k, k)$ () and $\lambda_1 \geq 2k - 1$, [*Ballantine–Orellana, 2006*]
- $\nu = (n - k, k)$, $\lambda = (n - r, r)$ [*Remmel–Whitehead*, 1994; *Blasiak–Mulmuley–Sohoni*,2013]
- $\nu = (n - k, 1^k)$ (), [*Blasiak 2012, Blasiak-Liu 2014*]
- Other special cases [Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova].

Geometric Complexity Theory Recap

VP vs VNP : permanent vs determinant

$$\mathsf{det}_n := \sum_{\sigma \in S_n} \mathsf{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)} \qquad \mathrm{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} x_{i,\sigma(i)}$$

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Conjecture (Valiant'78)

The (normalized) permanent $x_{11}^{n-m}\mathrm{per}_m \neq \det_n[A\mathbf{x}^T]$ *($n \times n$ determinant of affine linear forms in* $\{x_{ij}\}_{i,j=1}^{m}$*) for* $n = poly(m)$.

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GCT goal: If $\mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \subset \mathbb{C}[\overline{GL_{n^2}\mathsf{det}_n}]_d$, show that $n > poly(m)$.

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$$\mathbb{C}[\overline{GL_{n^2}\det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

Obstructions λ: if $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$ for $n > poly(m) \implies \mathrm{VP} \neq \mathrm{VNP}$.

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The (normalized) permanent $x_{11}^{n-m}\mathrm{per}_m \neq \mathrm{det}_n[Ax^T]$ ($n \times n$ determinant of affine linear forms in $\{x_{ij}\}_{i,j=1}^m$) for $n = poly(m)$.

GCT goal: If $\mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \subset \mathbb{C}[\overline{GL_{n^2}\mathrm{det}_n}]_d$, show that $n > poly(m)$.

$$\mathbb{C}[\overline{GL_{n^2}\mathrm{det}_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

Obstructions λ: if $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$ for $n > poly(m) \implies$ VP $\neq$ VNP.
If also $\delta_{\lambda,d,n} = 0$, then λ is an **occurrence obstruction**.

Conjecture (Mulmuley and Sohoni)

There exist occurrence obstructions that show $n > poly(m)$.

Geometric Complexity Theory Recap

VP vs VNP : permanent vs determinant

$$\det_n := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)} \qquad \mathrm{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} x_{i,\sigma(i)}$$

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Conjecture (GCT, Mulmuley and Sohoni)

There exist λ*, s.t.* $g(\lambda, n^d, n^d) = 0$ *and* $\gamma_{\lambda,d,n,m} > 0$ *for some* $n > poly(m)$.

GCT and Kronecker coefficients

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There exist λ, s.t. $g(\lambda, n^d, n^d) = 0$ and $\gamma_{\lambda,d,n,m} > 0$ for some $n > poly(m)$.

Theorem (Ikenmeyer-P)

Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n^d, n^d) = 0$(so $mult_\lambda \mathbb{C}[GL_{n^2} \det_n] = 0$), then $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m} per_m}] = 0$.

Theorem (Ikenmeyer-P)

For any ρ, let $n \geq |\rho|$, $d \geq 2$, $\lambda := (nd - |\rho|, \rho)$. Then $g(\lambda, n \times d, n \times d) \geq a_\lambda(d[n])$.

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Conjecture (Mulmuley and Sohoni 2001)

For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many m, there exists a partition λ occurring in $\mathbb{C}[\overline{GL_{n^2}X_{11}^{n-m}perm_m}]$ but not in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$, where $n = m^c$.

Theorem (Bürgisser-Ikenmeyer-P (FOCS'16, JAMS'18))

Let n, d, m be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If λ occurs in $\mathbb{C}[\overline{GL_{n^2}X_{11}^{n-m}perm_m}]$, then λ also occurs in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$. In particular, the Conjecture is false, there are no "occurrence obstructions".

No occurrence obstructions I: positive Kroneckers

Theorem (Ikenmeyer-Panova)

Let $n > 3m^4$, $\lambda \vdash nd$. *If* $g(\lambda, n \times d, n \times d) = 0$ *(so* $\mathrm{mult}_\lambda \mathbb{C}[\overline{GL_{n^2}\det_n}]_d = 0$*), then* $\mathrm{mult}_\lambda(\mathbb{C}[\overline{GL_{n^2}(x_{11})^{n-m}per_m)}]_d = 0$.

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Proof ingredients:

Theorem (Kadish-Landsberg)

If $\mathrm{mult}_\lambda \mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]_d > 0$, then $\lambda_1 \geq nd - md$ and $\ell(\lambda) \leq m^2$.

Theorem (Degree lower bound, [IP])

If $\lambda_1 \geq nd - md$ with $\gamma_{\lambda,d,n,m} > g(\lambda, n \times d, n \times d)$, then $d > \frac{n}{m}$.

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Theorem (Kronecker positivity, [IP])

If $\ell(\lambda) \leq m^2$, $\lambda_1 \geq nd - md$, $d > 3m^3$, and $n > 3m^4$, then $g(\lambda, n\times d, n\times d) > 0$, except for 6 special cases.

Kronecker positivity: squares, and decompositions

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If $\ell(\lambda) \leq m^2$, $\lambda_1 \geq nd - md$, $d > 3m^3$, and $n > 3m^4$, then $g(\lambda, n \times d, n \times d) > 0$, except for 6 special cases.

Proof sketch: decomposition + regrouping

$$(\lambda_2, \lambda_3, \ldots, \lambda_{m^2}) = \rho + \xi + \sum_{k=2}^{\ell} x_k((k-1) \times k) + \sum_{k=2}^{\ell} y_k((k-1) \times 2).$$

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Crucial facts:

- $g(k \times k, k \times k, k \times k) > 0$ [Bessenrodt-Behns].
- Transpositions: $g(\alpha, \beta, \gamma) = g(\alpha, \beta^T, \gamma^T)$ (with $\beta = \gamma = (w^h)$) [classical]

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- Hooks and exceptional cases: $g(\lambda, w^h, w^h) > 0$ for all $\lambda = (hw - j - |\rho|, 1^j + \rho)$ for $|\rho| \leq 6$ and almost all js. [Ikenmeyer-Panova'16]

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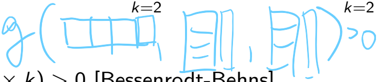

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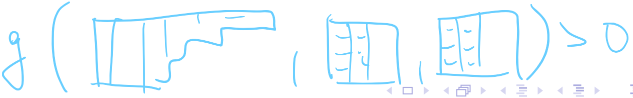
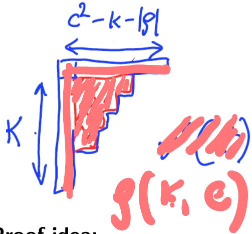
- Semigroup property:
  $g(\alpha^1 + \alpha^2, \beta^1 + \beta^2, \gamma^1 + \gamma^2) \geq \max(g(\alpha^1, \beta^1, \gamma^1), g(\alpha^2, \beta^2, \gamma^2)$. [classical]



Kronecker positivity I: hook-like λs

Proposition (Ikenmeyer-P)



Fix a partition ρ. For $k \geq \ell(\rho)$ and c, such that $c^2 \geq k+|\rho|+\rho_1+1$ define $\rho(k,c) := 1^{k+1}+(c^2-k-|\rho|-1,\rho)$. Suppose that there exists an integer a and sets $H^1, H^2 \subset [\ell, 2a+1]$, such that

$$g(\rho(k,a), a \times a, a \times a) > 0 \text{ for all } k \notin H^1 \cup (a^2 - H^2)$$

Then

$$g(\rho(k,b), b\times b, b\times b) > 0 \text{ for all } b \geq a \text{ and } k \notin H^1 \cup (b^2 - H^2$$

Proof idea:
Kronecker symmetries and semigroup properties:
Let $P_c = \{k : g(\rho(k,c), c \times c, c \times c) > 0\}$, we have
Claim: Suppose that $k \in P_c$, then $k, k+2c+1 \in P_{c+1}$.

Kronecker positivity I: hook-like λs

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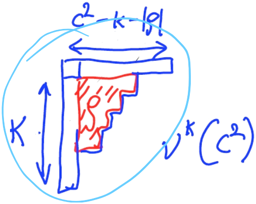

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Corollary

We have that $g(\lambda, h \times w, h \times w) > 0$ for $\lambda = (hw - j - |\rho|, 1^j + \rho)$ for most "small" partitions ρ and all but finitely many values of j.

Kronecker vs plethysm via complexity

Plethysm coefficients: $a_\lambda(d[n]) = \mathrm{mult}_\lambda S^d(S^n V)$ ($GL_N(V)$ representations)

Kronecker coefficients: $g(\lambda, n^d, n^d) = \mathrm{mult}_\lambda(\mathbb{S}_{n^d} \otimes \mathbb{S}_{n^d})$ (S_n representations)

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Fix ρ, and let $n \geq |\rho|$. Let $\lambda = (nd - |\rho|, \rho)$. Then $g(\lambda, n \times d, n \times d) \geq a_\lambda(d[n])$.

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Let $V_m := Sym^m \mathbb{C}^{m^2}$ (polynomials in m^2 variables of degree m).

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KL'14: If $\mu \vdash md$ then $\mathrm{mult}_{\mu+d(n-m)}(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m}V_m}]_d \geq a_\mu(d[m])$.

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Stability: $g(\lambda, n \times d, n \times d) = g(\mu, m \times d, m \times d)$ [Manivel'98]

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then $dc(f_m) > n$ for some $f_m \in V_m$.

$$\Longrightarrow \mathrm{mult}_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m}V_m})]_d \geq a_\mu(d[m]) = a_\lambda(d[n]) > g(\lambda, n \times d, n \times d)$$
$$> \delta_{\lambda,d,n} \Longrightarrow \max_{f \in V_m} dc(f) > n \to \infty$$

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Plethysm coefficients: $a_\lambda(d[n]) = \mathrm{mult}_\lambda S^d(S^n V)$ ($GL_N(V)$ representations)

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Proposition (Ikenmeyer-P)

Fix ρ, and let $n \geq |\rho|$. Let $\lambda = (nd - |\rho|, \rho)$. Then $g(\lambda, n \times d, n \times d) \geq a_\lambda(d[n])$.

Proof: $\lambda = \mu + d(n-m)$. Suppose $g(\lambda, n \times d, n \times d) < a_\lambda(d[n])$.
Let $V_m := Sym^m \mathbb{C}^{m^2}$ (polynomials in m^2 variables of degree m).

KL'14: If $\mu \vdash md$ then $\mathrm{mult}_{\mu+d(n-m)}(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m} V_m}]_d \geq a_\mu(d[m])$.

Stability: $g(\lambda, n \times d, n \times d) = g(\mu, m \times d, m \times d)$ [Manivel'98]

GCT: If $\mathrm{mult}_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m} V_m})]_d \geq g(\lambda, n \times d, n \times d)$ ($\geq \mathrm{mult}_\lambda \mathbb{C}[\overline{GL_{n^2} \det_n}]_d$)
then $dc(f_m) > n$ for some $f_m \in V_m$.

$$\implies \mathrm{mult}_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m} V_m})]_d \geq a_\mu(d[m]) = a_\lambda(d[n]) > g(\lambda, n \times d, n \times d)$$
$$> \delta_{\lambda,d,n} \implies \max_{f \in V_m} dc(f) > n \to \infty$$

Multiplicity obstructions vs occurrence obstructions

Toy problem: Factor $\ell_1^n + \cdots + \ell_k^n$ into linear forms? $(k > 2)$

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If $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ch}_m^n]_d) < \mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ps}_{m,k}^n]_d)$ – λ is a *multiplicity obstruction*.
If also $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ch}_m^n]_d) = 0$, then λ is called an *occurrence obstruction*.

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Theorem (Dörfler–Ikenmeyer-P, ICALP'19, SIAGA'20)

General: *Let* $m \geq 3$, $n \geq 2$. *We have* $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ch}_m^n]_{n+1}) < \mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ps}_{m,n+1}^n]_{n+1})$ *for* $\lambda = (n^2 - 2, n, 2)$, *i.e.,* λ *is a multiplicity obstruction that shows* $\mathsf{P}_{m,n+1}^n \not\subseteq \mathsf{Ch}_m^n$.
Finite: *Stronger separation: When* $k = 4$, $n = 6$, $m = 3$, $d = 7$, *or when* $k = 4$, $n = 7$, $m = 4$, $d = 8$, *for* $\lambda = (n^2 - 2, n, 2)$ *we have* $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ch}_m^n]_d) < \mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ps}_{m,k}^n]_d)$.
No occurrence obstructions, *for all* k*: for all* m*-partitions* μ *that satisfy* $a_\mu(d[n]) > 0$ *we have* $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ch}_m^n]_d) > 0$ *in the finite settings.*

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[DIP'19]: $m \geq 3$, $n \geq 2$, $k = d = n + 1$, $\lambda = (n^2 - 2, n, 2)$. We have $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ch}_m^n]_d) < \mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ps}_{m,k}^n]_d)$. When $k = 4$, $n = 6$, $m = 3$, $d = 7$, or when $k = 4$, $n = 7$, $m = 4$, $d = 8$, for $\lambda = (n^2 - 2, n, 2)$ we have $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ch}_m^n]_d) < \mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ps}_{m,k}^n]_d)$ and no occurrence obstructions.
Proof:

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- [Bürgisser-Ikenmeyer-P] $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ps}_{m,k}^n]_d) = a_\lambda(d[n])$ for $k \geq d$ or $(k, n, m, d) \in \{(4, 6, 3, 7), (4, 7, 4, 8)\}$.

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- No occurrence obstructions for $(n,m) = (6,3),(7,4)$: if $a_\lambda(d[n]) > 0$ then $\mathrm{mult}_\lambda(\mathbb{C}[\mathsf{Ch}_m^n]_d > 0$. (verified by direct heavy computer calculation)

Plethysms and Foulkes conjecture

$$\binom{a}{b}_q = \frac{\prod_{i=1}^{a}(1-q^i)}{\prod_{i=1}^{b}(1-q^i)\prod_{i=1}^{a-b}(1-q^i)}$$

Proposition (Dörfler-Ikenmeyer-P'19)

The plethysm coefficient for $\lambda = (L, r, 2)$ is equal to

$$a_\lambda(d[n]) =$$

$$(q^{r+1}) @ \binom{n+d-2}{n}_q \frac{q(1-q^n)(1-q^2+q-q^n)}{1-q^2}$$

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Corollary

Let $d = n+1$ and $\lambda = (n^2+n-2-r, r, 2)$. Then $a_\lambda((n+1)[n]) - a_\lambda(n[n+1]) \geq 0$, with

$$a_\lambda((n+1)[n]) - a_\lambda(n[n+1]) = \begin{cases} 0, & \text{when } r < n, \\ 1, & \text{when } r = n, \\ > 0, & \text{when } r > n \text{ and } n \geq 7, \end{cases}$$

with the exception in the last case when $n = 8$, and $r = 35$ when $a_{(35,35,2)}(9[8]) = a_{(35,35,2)}(8[9])$.

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Corollary: Foulkes conjecture holds for $\lambda = (ab - r - 2, r, 2)$ and $a = b + 1$.

Formulas

Schur function $s_\lambda[x_1y_1, x_1y_2, \ldots, x_2y_1, \ldots]$

$$= \sum g(\lambda,\mu,\nu)\, s_\mu(x)\, s_\nu(y)$$

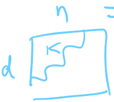
$a_\lambda(n[d])$:

$$h_n[h_d] = h_n(x_1^d, x_1^{d-1}x_2, x_1^{d-1}x_3, \ldots, x_1^{d-2}x_2^2, \ldots)$$

$$= \sum_{\lambda \vdash nd} a_\lambda(n[d])\, s_\lambda(x_1, x_2, \ldots)$$

$$a_{(nd-k,k)}(n[d]) = \frac{x_1^{nd-k+1}x_2^k - x_2^{nd-k+1}x_1^k}{x_1 - x_2}$$

$$= \text{coef of } (x_1^{nd-k+1}x_2^k) \text{ at } (x_1 - x_2)\, h_n(x_1^d, x_1^{d-1}x_2, \ldots, x_2^d)$$



$$= p_k(n,d) - p_{k-1}(n,d)$$

Thank you!