Demystifying the border of depth-3 circuits

Accepted to the 62^{nd} IEEE Symposium on Foundations of Computer Science (FOCS 2021).

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24^{th} January, 2022

School and Conference on Geometric Complexity Theory
Jan 2022, Chennai (virtual)

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Algebraic Complexity Theory

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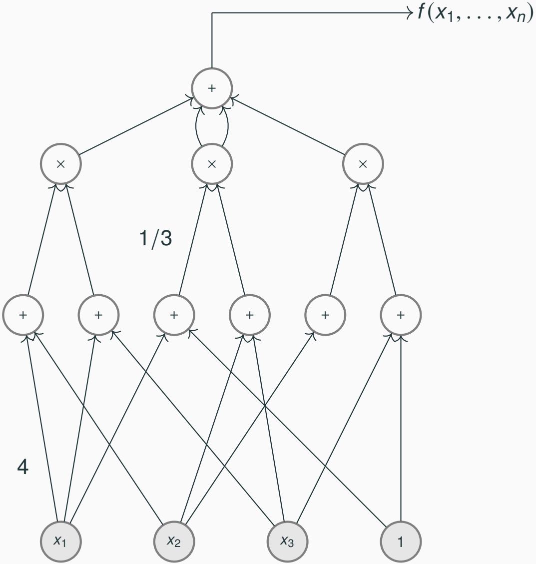
- [$\mathsf{P} \stackrel{?}{=} \mathsf{NP}$, Aaronson 2011] calls GCT *"The String Theory of Computer Science"*.

Algebraic circuits



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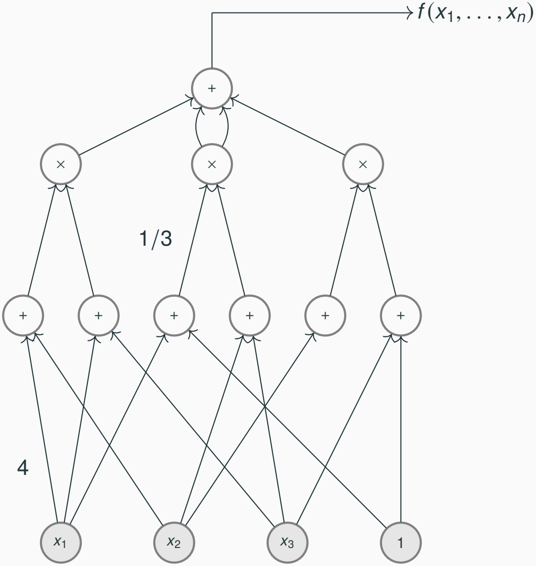

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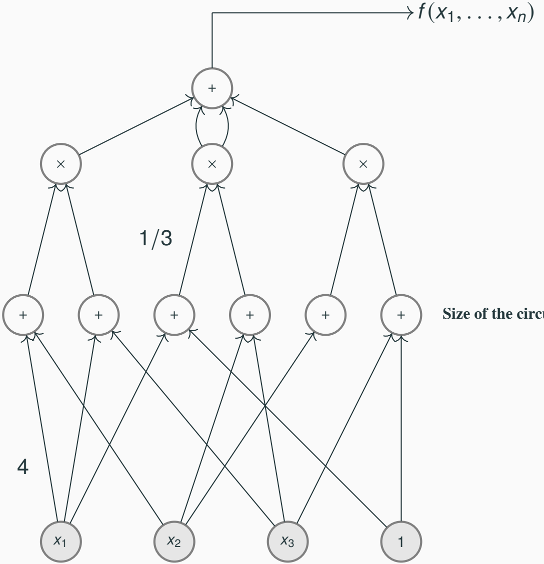

Size of the circuit = number of nodes + edges

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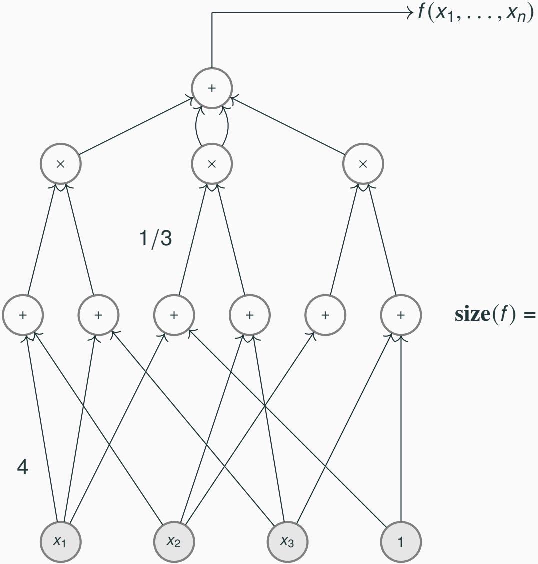

size(f) **= min size of the circuit computing** f

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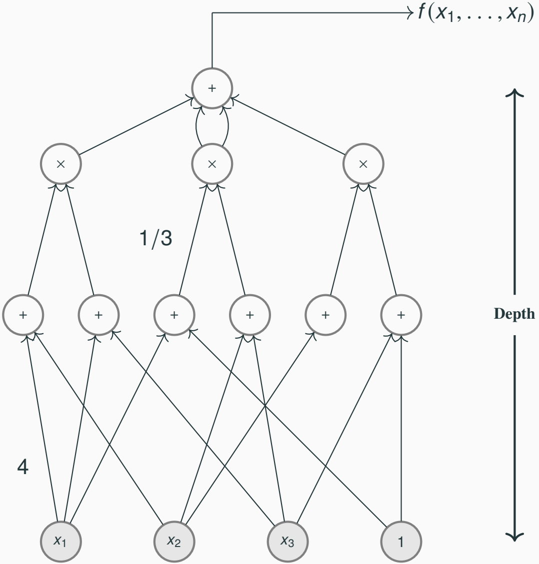

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The class VP is defined as the set of all sequences of polynomials $(f_n(x_1,\ldots,x_n))_{n\geq 1}$ such that $\mathrm{size}(f_n), \deg(f_n)$ are both bounded by n^c for some constant c.

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- E.g. $\mathrm{dc}(x_1 \cdots x_n) = n$, since

$$x_1 \cdots x_n = \det \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & x_n \end{pmatrix} .$$

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- Connections: Linear algebra, Volume, counting planar matchings.

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$\mathsf{VBP} \neq \mathsf{VNP}$ & $\mathsf{VP} \neq \mathsf{VNP}$. Equivalently, $\mathsf{dc}(\mathsf{perm}_n)$ and size(perm_n) are both $n^{\omega(1)}$.

[Also, $\mathsf{VBP} \neq \mathsf{VP}$. A candidate?]

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  - Assuming GRH (Generalized Riemann hypothesis), the results hold over $\mathbb{C}$ as well.

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- Can there be 'algebraic natural proofs' to prove $\mathsf{VP} \neq \mathsf{VNP}$? Some answers: [Chatterjee-Kumar-Ramya-Saptharishi-Tengse 2020, Kumar-Ramya-Saptharishi-Tengse 2020].

Border Complexity and GCT

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- $\mathsf{WR}(h) \leq r$ is denoted as $h \in \Sigma^{[r]} \wedge \Sigma$ (homogeneous *depth-3 diagonal* circuits).

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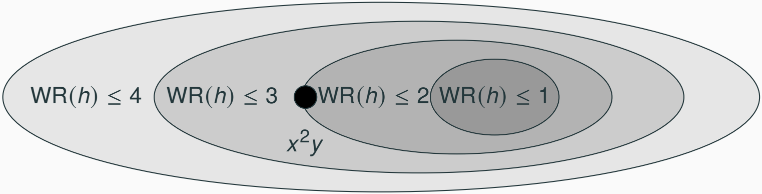

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- The subtlety is *gone*: $X_n := \{h \mid \overline{\mathsf{WR}}(h) \leq n\}$, is now a **closed** set.
- On to proving lower bounds: To show $\overline{\mathsf{WR}}(p) > n$, for some p, it suffices to show that $p \notin X_n$, i.e. find a *continuous* function f that vanishes on X_n but not on p.

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  Border tensor rank is *directly* related to the matrix multiplication exponent ω [Bini 1980, Coppersmith-Winograd 1990].

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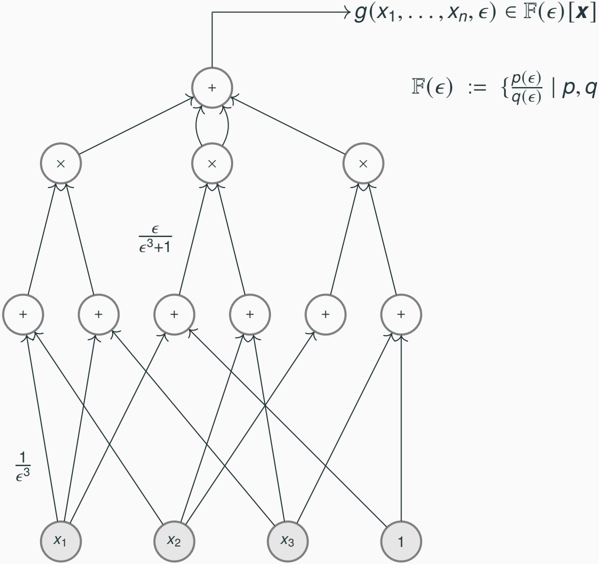
- Yes! Via *'approximative circuits'*.

Approximative circuits (continued)



$$\mathbb{F}(\epsilon) \;:=\; \left\{\frac{p(\epsilon)}{q(\epsilon)} \mid p, q \in \mathbb{F}[\epsilon], q(\epsilon) \neq 0\right\}$$

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- Suppose, we assume the following:
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- Summary: g_0 is really something **non-trivial** and being 'approximated' by the circuit since $\lim_{\epsilon \to 0} g(\boldsymbol{x}, \epsilon) = g_0$.

Algebraic approximation (continued)

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- Both det and perm have 'nice' **symmetries**.
- Symmetry-characterization **avoids** the Razborov–Rudich barrier: *Very few* functions are symmetry-characterized, so symmetry-characterization violates the largeness criterion!

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- Upper bounds and lower bounds become more algebro-**geometric** in nature.
- Further potential applications in identity testing and understanding its 'robustness'.

Border depth-3 circuits

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- How powerful are $\Sigma^{[k]}\Pi^{[d]}\Sigma$ circuits? Are they *universal*?
- **No**. E.g. the *Inner Product* polynomial $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1y_1 + \ldots + x_{k+1}y_{k+1}$ **cannot** be written as a $\Sigma^{[k]}\Pi^{[d]}\Sigma$ circuit, *regardless* of the product fanin d!

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Border depth-3 fanin 2 circuits are 'universal' [Kumar 2020]

Let P be *any* homogeneous n-variate degree d polynomial. Then, $P \in \overline{\Sigma^{[2]}\Pi^{[D]}\Sigma}$, where $D := \exp(n, d)$.

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4. Divide by ϵ^d and rearrange to get

$$P + \epsilon^d \cdot R(\boldsymbol{x}, \epsilon) = -\epsilon^{-d} + \epsilon^{-d} \cdot \prod_{i=1}^{m}\prod_{j=1}^{d}(\alpha_j + \epsilon \cdot \ell_i) \in \Sigma^{[2]}\Pi^{[md]}\Sigma .$$

$\square$

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Remark. The result holds if one replaces the top-fanin-2 by arbitrary constant k.

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- *Infinitely* many factorizations may give *infinitely* many limits.

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- We devise a technique called DiDIL –
  Divide, **D**erive, **I**nterpolate/ Induct with **L**imit.

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$$\begin{aligned} f + \epsilon \cdot S &= T_1 + T_2 \\ \implies \quad \Phi(f) + \epsilon \cdot \Phi(S) &= \Phi(T_1) + \Phi(T_2) \\ \implies \quad \Phi(f)/\tilde{T}_2 + \epsilon \cdot \Phi(S)/\tilde{T}_2 &= \Phi(T_1)/\tilde{T}_2 + \epsilon^{a_2} \\ \implies \quad \partial_z\left(\Phi(f)/\tilde{T}_2\right) + \epsilon \cdot \partial_z\left(\Phi(S)/\tilde{T}_2\right) &= \partial_z\left(\Phi(T_1)/\tilde{T}_2\right) =: g_1 \, . \end{aligned} \tag{1}$$

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- Here Σ means just a linear polynomial ℓ.

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- Suffices to compute $\lim_{\epsilon \to 0} g_1 \mod z^d$.

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Thus,

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Derandomizing border depth-3 circuits

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  - Graph Theory [Lovasz'79], [Fenner-Gurjar-Theirauf'19]
  - Primality Testing [Agrawal-Kayal-Saxena'04].

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Theorem 2 (Derandomizing polynomal-sized depth-3 top-fanin-k circuits) [Dutta-Dwivedi-Saxena 2021]

There exists an explicit quasipolynomial-time ($s^{O(\log\log s)}$) hitting set for size-s $\overline{\Sigma^{[k]}\Pi\Sigma}$ circuits, for any constant k.

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