(paper on arxiv)

\[ f(x_1, \ldots, x_m) = \sum_{i=1}^n l_i(x_{i1}, \ldots, x_{im})^d \quad \text{deg}(f) = d. \]

Smallest possible \( n \): Waring rank of \( f \).

When \( d, m \to \infty \): exponential lower bounds.

Method: partial derivatives.

This talk: \( d = 3 \). Wanted: \( \Omega(m^2) \) lower bound (worst case).

Current lower bounds: \( O(m) \).

Barren result: Rank method cannot go beyond \( O(m) \).

\[ l_i \text{ if } \text{IR}(x_{i1}, \ldots, x_{im}). \]
\[
T = \frac{1}{2} \sum_{i,j} \text{Tr} \left( T_{ij} \otimes \mu_2 \right)
\]

where \( \mu_2 = \text{diag}(-1, 1) \).

The slice: \( T = (16) \).

Proposed approach: method of commuting extensions (Dagani-Schiff, Tannor'85).
A matrix problem: commuting extensions.

Given square matrices \( A_1, \ldots, A_k \), we want to extend them to obtain commuting matrices.

\[
\begin{bmatrix}
 X_1 & \cdots & X_k
\end{bmatrix}
\begin{bmatrix}
 \mathbf{A}_1 & \mathbf{B}_1 \\
 \mathbf{C}_1 & \mathbf{D}_1
\end{bmatrix}
\begin{bmatrix}
 Y_1 \\
 \vdots \\
 Y_k
\end{bmatrix}
\]

What is the smallest possible \( n \)?

**Lemma (Strassen'83):** \( \frac{1}{2} \text{rank}(A_2 A_2 - A_2 A_2) \geq n \geq m + \frac{1}{2} \text{rank}(A_2 A_2 - A_2 A_2) \).

**Theorem (Strassen'83):**

For a tensor \( T \) with 3 slices \( A_2, A_2, A_3 \):

\[
\text{rank}_k(T) \geq m + \frac{1}{2} \text{rank}(A_2 A_2^{-1} A_3 - A_2 A_2^{-1} A_2).
\]
Constructions of commuting extensions:

1) \[ X_i = \left( \frac{A_i}{A_i} - A_i \right) \] shows that \( n \leq 2m \) due to Jeroen Zuiddam.

\[ X_i X_j = 0. \]

2) For \( L \) matrices: \( X_a = \begin{pmatrix} A_a & A_a \\ A_a & A_a \end{pmatrix} \), \( X_e = \begin{pmatrix} A_e & A_e \\ A_e & A_e \end{pmatrix} \), symmetric if \( A_a, A_e \) symmetric.

3) For \( h \) matrices: \( X_h = \begin{pmatrix} A_h & i A_h \\ i A_h & -A_h \end{pmatrix} \) if \( i = -1 \)

\[ X_h X_h = 0. \]
Connection to algebraic complexity

Theorem 1: Let \( T \in S^3 \mathbb{R}^n \) with symmetric rank \( \leq n \). There exists a set \( S \) of order 3 and size \( n+m \) such that:

(i) The slices of \( S \) commute.

(ii) \( T \) is a subarray of \( S \): \( T_{ijk} = S_{ijk} \) for \( i,j,k \leq m \).

Contrapositive: lower bound on extension size

\[ \Rightarrow \] lower bound on \( n \).

What could go wrong?

1) proving LB on size of commuting extension is hard.

2) Perhaps \( T \) has high rank, but a small commuting extension is impossible.
\[ \sum \text{area of } S \text{ as symmetric (T)} \}

Application of Theorem: If \( \exists w \text{ (arg of } S) \),

\[ \frac{i=n}{F = 1 \text{ m}^2} \]

From \( \text{horizontal decomposition} \),

The degree of \( S \) component \( \Rightarrow S \text{ has an odd order} \).

Theorem: Let \( S \) be a nodal symmetric beam, \( S \in G \).

Theorem: If \( (i) + (ii) \) hold then,

\( \sum \text{area of } S \text{ as symmetric (T)} \),
Theorem 3: $T$ is of symmetric rank $\leq n$ iff there is a real symmetric tensor $S$ of size $n \times n$ such that:

(i) The slices of $S$ commute.

(ii) $T_{i,j,k} = S_{i,j,k}$ for $1 \leq i, j, k \leq n$.

(iii) Any matrix in the subspace spanned by the slices of $S$ is of rank $\leq n$. 
