On tensor rank and commuting matrices

(paper on arxiv)

$$f(x_1, \ldots, x_m) = \sum_{i=1}^{r} l_i(x_1, \ldots, x_m)^d \qquad \deg(f) = d.$$

Smallest possible r: Waring rank of f.

When $d, m \to +\infty$: exponential lower bounds.

Method: partial derivatives.

This talk: $d = 3$. Wanted: $\Omega(m^2)$ lower bound (worst case).

Current lower bounds: $O(m)$.

Barrier result: Rank method cannot go beyond $O(m)$.

$l_i, f \in \mathbb{R}[x_1, \ldots, x_m]$.

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Tensor point of view: $f(x_1, \dots, x_n) = \sum_{i,j,k} T_{ijk} x_i x_j x_k$.

$T = (T_{ijk})_{1 \leq i,j,k \leq n}$ is a symmetric tensor: $T_{jik} = T_{ijk} = T_{ikj} \dots$

$T = \sum_{i=1}^{n} u_i^{\otimes 3}$, where $l_i(x) = \langle u_i, x \rangle$.

k-th slice: $T_k = (T_k)_{ij} = T_{ijk}$. it is a real symmetric matrix.

<u>Proposed approach:</u> method of commuting extensions.

(name borrowed from: Degani - Schiff - Tannor '05)

A matrix problem: commuting extensions.

Given square matrices $A_1, \dots, A_k$, we want to extend them to obtain commuting matrices.

$$X_i = \underbrace{\left(\begin{array}{c|c} \overbrace{A_i}^{m} & B_i \\ \hline C_i & D_i \end{array}\right)}_{n}$$

$X_i X_j = X_j X_i$. What is the smallest possible n?

For $k=2$:

Lemma (Strassen '83): $n \geq m + \frac{1}{2} \operatorname{rank}(A_1 A_2 - A_2 A_1)$.

$n \in [m, 3\frac{m}{2}]$.

Theorem (Strassen '83):

For a tensor T with 3 slices A_1, A_2, A_3:

$$\operatorname{rank}(T) \geq m + \frac{1}{2} \operatorname{rank}(A_2 A_1^{-1} A_3 - A_3 A_1^{-1} A_2).$$

Constructions of commuting extensions:

1) $X_i := \begin{pmatrix} A_i & -A_i \\ A_i & -A_i \end{pmatrix}$ shows that $n \leq 2m$ due to Jeroen Zuiddam.

$X_i X_j = 0.$

2) For 2 matrices: $\underbrace{X_1 := \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}, \quad X_2 := \begin{pmatrix} A_2 & A_1 \\ A_1 & A_2 \end{pmatrix}}_{\text{symmetric if } A_1,\, A_2 \text{ symmetric.}}$

3) For k matrices: $X_j := \begin{pmatrix} A_j & iA_j \\ iA_j & -A_j \end{pmatrix}$ $\quad i^2 = -1$

$X_j X_k = 0.$

Connection to algebraic complexity

Theorem 1: Let $T \in S^3 \mathbb{R}^m$ with symmetric rank $\leq r$.
There exists S of order 3 and size $r+m$ such that:

(i) The slices of S commute.

(ii) T is a subtensor of S: $T_{ijk} = S_{ijk}$ for $i,j,k \leq m$.

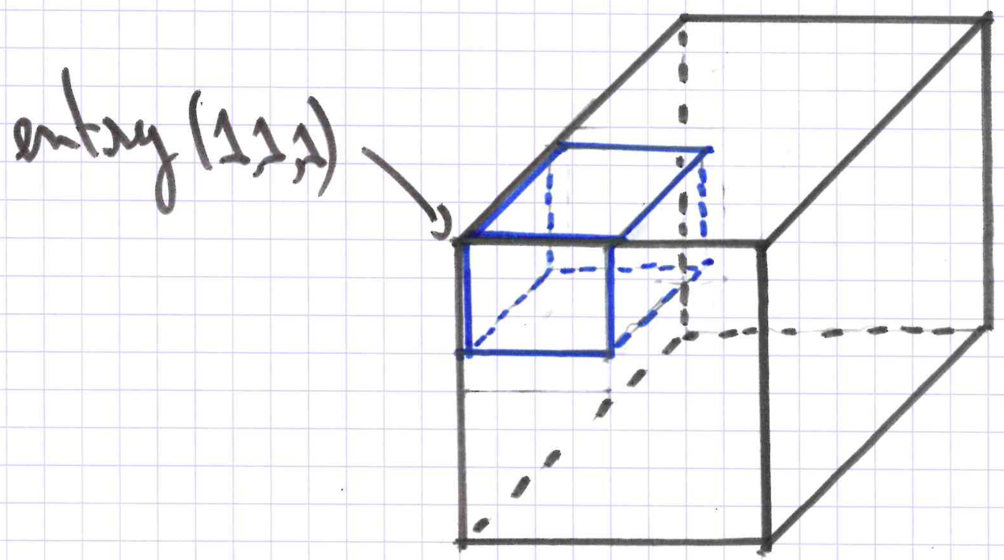
Contrapositive: lower bound on extension size
$\Rightarrow$ lower bound on r.

What could go wrong?

1) proving LB on size of commuting extensions is hard.

2) Perhaps T has ~~high rank, but a small~~ commuting extension?

IMPOSSIBLE.



$\mathcal{S}$: size $n+1$.

T: size n.

Theorem 2: If (i) + (ii) hold then sym. rank$(T) \leq \underbrace{m+n}_{\text{size of } S}$.

Theorem 4: Let S be a real symmetric tensor, of size m and order 3.

The slices of S commute $\iff$ S has an orthogonal decomposition.

$$S = \sum_{i=1}^{k} u_i^{\otimes 3} \quad \text{where } u_i \perp u_j \text{ for } i \neq j.$$

$(\text{so } k \leq m)$.

Application to Theorem 2: $k \leq m+n$ (size of S).

T is a subtensor of S, so sym rank$(T) \leq$ sym rank(S) $\leq m+n$.

Theorem 3: T is of symmetric rank $\leq r$ iff there is a real symmetric tensor S of size $r+n$ such that:

(i) The slices of S commute.

(ii) $T_{ijk} = S_{ijk}$ for $1 \leq i, j, k \leq n$.

(iii) Any matrix in the subspace spanned by ~~the~~ the slices of S of size $n+r$. is of rank $\leq r$.