gct2022 : School and Conference on Geometric Complexity Theory

17-28 Jan 2022 Chennai (India)

Algebraic Complexity: Structural results

Depth reduction, Homogeneization, Multilinearization, ...

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Overview

1. Why to look at structural results
2. Homogeneization / (Set)-multilinearization
   - Homogeneization
   - Multilinearization
3. Parallelization
   - Classical depth reductions of [Brent] and [VSBR]
   - To constant depth

General roadmap for lower bounds

Four steps in most lower bound proofs

Step 1: Finding a "nice" form for the model

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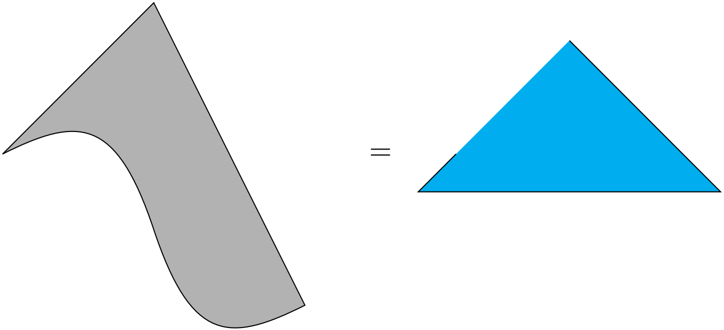

Meta Theorem 1

Every small circuit can be equivalently computed as a "nice" 

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Step 2: Constructing a complexity measure

Meta Theorem 2

Find a map $\Gamma : \mathbb{F}[\mathbf{x}] \to \mathbb{Z}_{\geq 0}$ such that Γ () is small.

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Step 3: Heuristic estimate for a random polynomial

Meta Theorem 2

Convince yourself that $\Gamma(R)$ *must be* LARGE for a random polynomial R.

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Every small circuit can be equivalently computed as a "nice" $\triangle$

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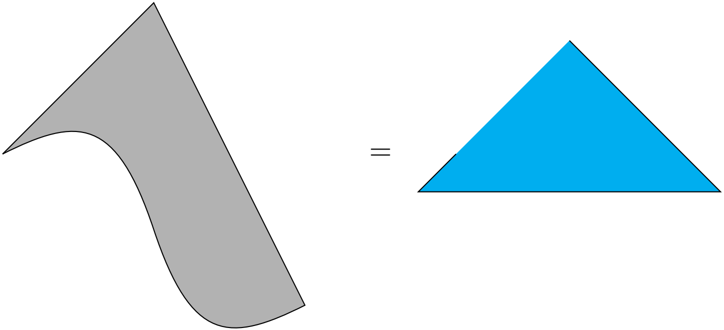
Step 4: Find a hay in the haystack

Four steps in most lower bound proofs

Step 1: Finding a "nice" form for the model



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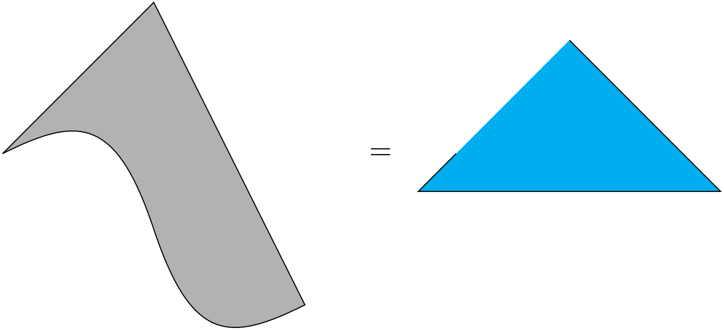

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Homogeneization, (Set)-multilinearization, Depth reduction

Arithmetic models

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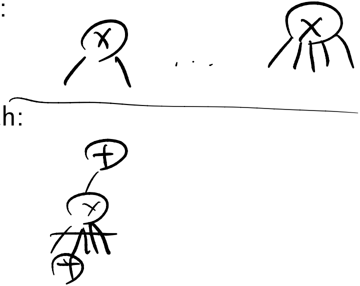
- Reverse inclusions?
- Circuit of size s $\rightsquigarrow$Formula of size $s^{O(\log d)}$.

Few words about fan-ins

If nothing is mentionned

- For circuits, formula of "large depth":
  - $+$-gate : unbounded
  - $*$-gate : constant
- For circuits, formula of constant depth:
  - $+$-gate : unbounded
  - $*$-gate : unbounded



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(Det_n) in $\Sigma\Pi\Sigma\Pi$

* non hom $\to n^{O(\sqrt[3]{n})}$
* hom $\to 2^{\Omega(\sqrt{n})}$

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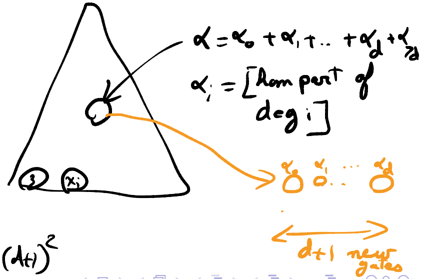
$$g = \sum_j g_j \longrightarrow g^{(i)} = \sum_j g_j^{(i)}$$

$$g = g_1 \times g_2 \longrightarrow g^{(i)} = \sum_{j=0}^{i} g_1^{(j)} \times g_2^{(i-j)}$$



Total: $s \mapsto s(d+1) + s(d+1) \times (d+1) \le s(d+1)^2$

(Syntactic) (Set)-multilinearization

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- Semantic vs. $\boxed{\text{Syntactic}}$

A circuit syntac set-multilinear

* $(c) \to \{\ \}$
* $(x_i) \to \{j\}$

$X_j \supset$

* $(+)$ with children $\alpha_1 \ldots \alpha_p$ $\longmapsto S_{\alpha_1} = S_{\alpha_2} = \ldots = S_{\alpha_p} = S_\alpha$
* $(\times)$ with children α_1, α_2 $\longmapsto S_\alpha = S_{\alpha_1} \uplus S_{\alpha_2}$.

A circuit is called synt multilinear

* $(c) \to \{\ \}$
* $(x_i) \to \{x_i\}$
* $(+)$ with children (α), (β); S_α, S_β $\longmapsto S_\alpha \cup S_\beta$
* $(\times)$ with children (α), (β) $\longmapsto S_\alpha \uplus S_\beta$

(Syntactic) (Set)-multilinearization

- Multilinear, Set-multilinear
- Semantic vs. Syntactic
- Expensive!

	Syn. Multilinear	Syn. Set-multilinear
Ciruits	???	$s \cdot 2^{O(d)}$
Formulas	???	$2^{O(d \log\log s)}$
Hom. formulas of cst depth	???	$s \cdot d^{O(d)}$

$\rightsquigarrow s^{\log d} 2^{O(d \log d)}$

Trivial $\mapsto 2^n$

A short history of depth reduction

Class	Depth	Size	
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	4^*	$s^{O(d^{1/3})}$	[Gupta-Kamath-Kayal-Saptharishi]
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Other depth reductions in lower bounds

Multilinear formulas [Raz, Raz-Yehudayoff]

$$f \quad = \quad \sum_{i=1}^{s} g_{i1} \cdot g_{i2} \dots g_{i\ell} \quad , \quad (1/3)^j \cdot n \ \leq \ \mathrm{Var}(g_{ij}) \ \leq \ (2/3)^j \cdot n$$

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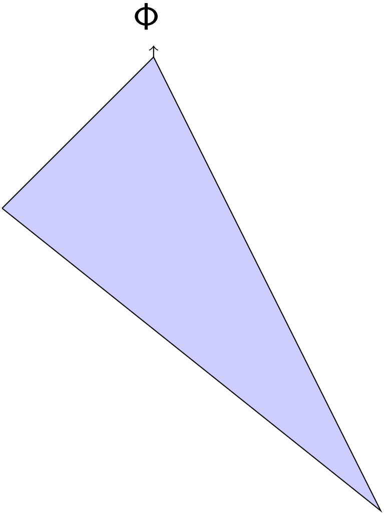
Homogeneous formulas [Hrubes-Yehudayoff]

$$f \quad = \quad \sum_{i=1}^{s} g_{i1} \cdot g_{i2} \ldots g_{i\ell} \quad , \quad (1/3)^j \cdot d \;\leq\; \deg(g_{ij}) \;\leq\; (2/3)^j \cdot d$$

Depth reducing formulas



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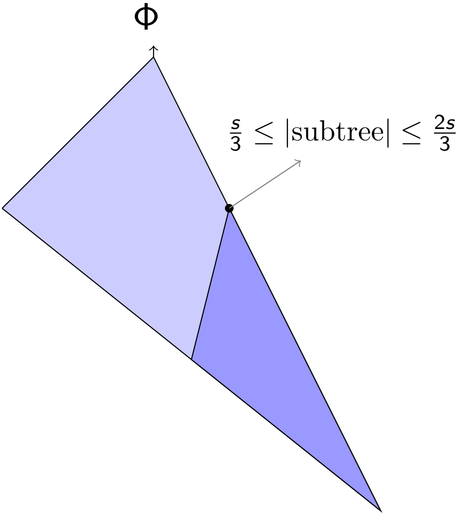

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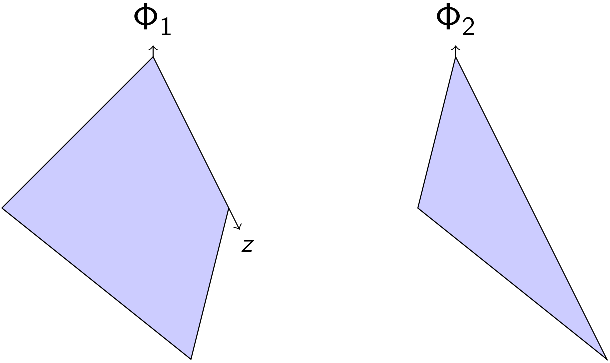

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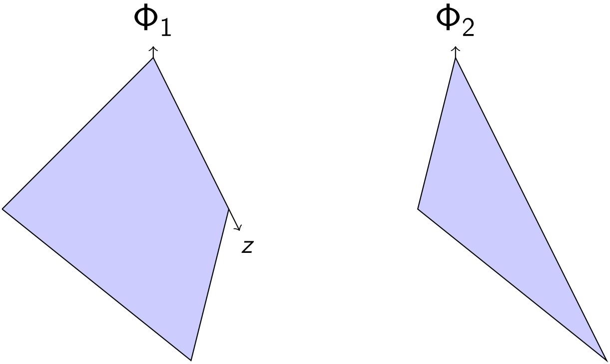
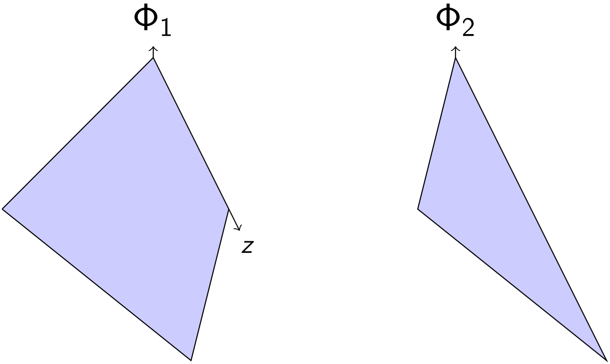


$$\begin{array}{lllll} \Phi_1(z) & = & A \cdot z & + & B \\ \Phi & = & A \cdot \Phi_2 & + & B \end{array}$$

Depth reducing formulas



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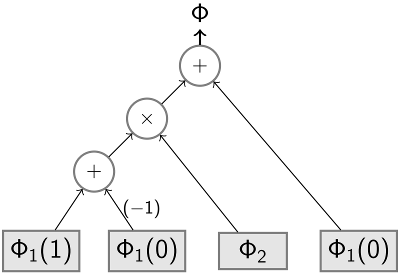

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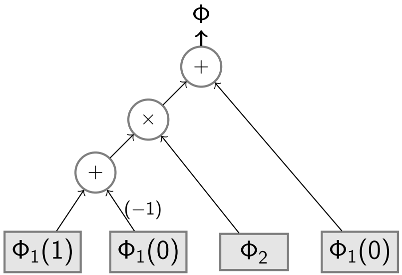
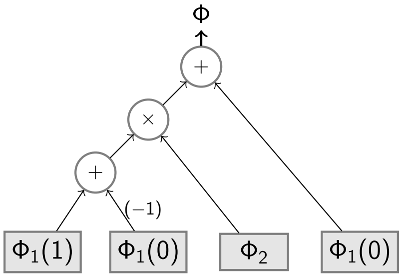


$$\mathrm{Size}(s) \leq \quad 4 \cdot \mathrm{Size}(2s/3) \quad + \quad O(1)$$
$$\mathrm{Depth}(s) \leq \quad \mathrm{Depth}(2s/3) \quad + \quad O(1)$$

Depth reducing formulas



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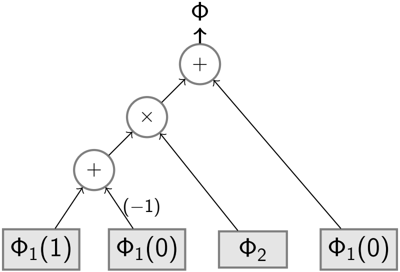

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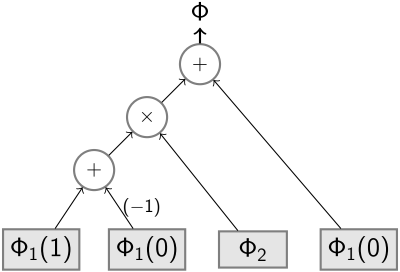

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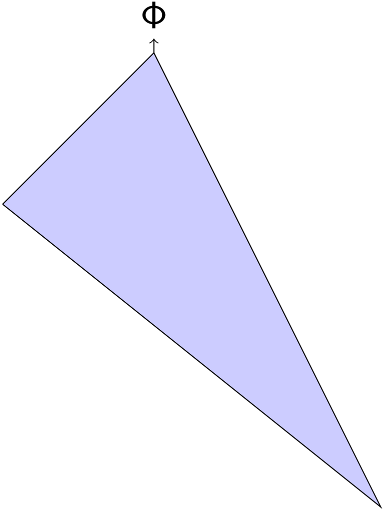
Adapting to circuits



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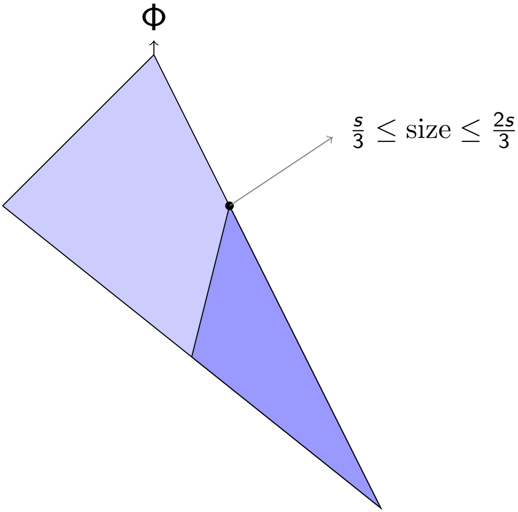

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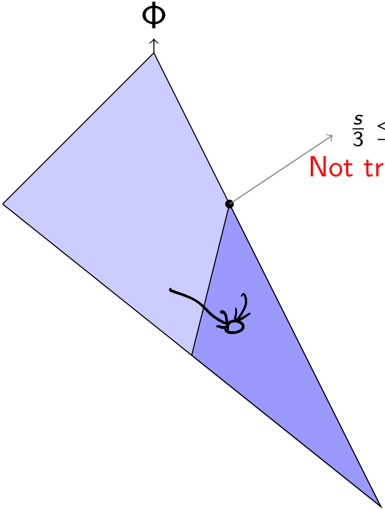
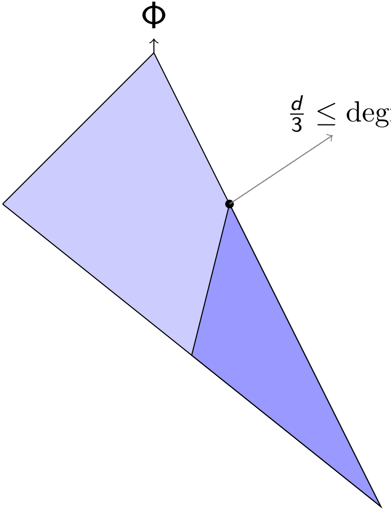


$\frac{s}{3} \leq \text{size} \leq \frac{2s}{3}$

Not true for circuits!

Adapting to circuits



$\frac{d}{3} \leq \mathrm{degree} \leq \frac{2d}{3}$ (start with a homogeneous circuit)

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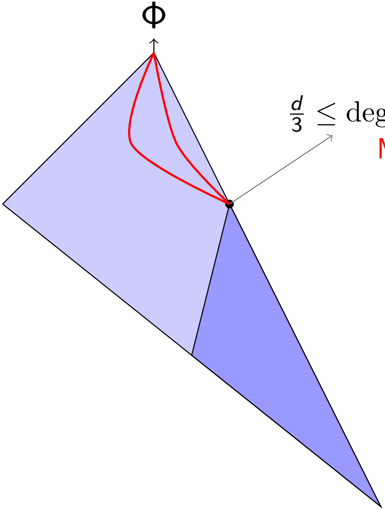

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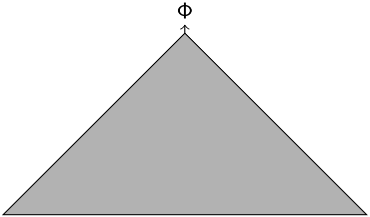
Multiple paths from root!

Adapting to circuits: Attempt 1



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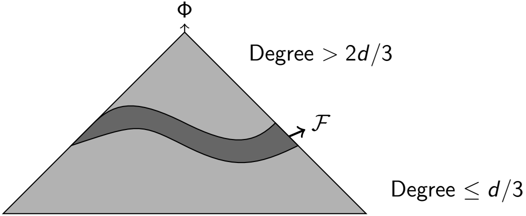

$$\mathcal{F} \quad = \quad \left\{ v \in \Phi \mid \frac{d}{3} < \deg(v) \leq \frac{2d}{3} \right\}$$

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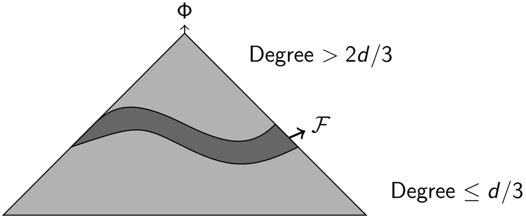

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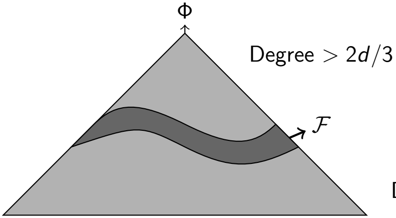

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each have degree at most $2d/3$

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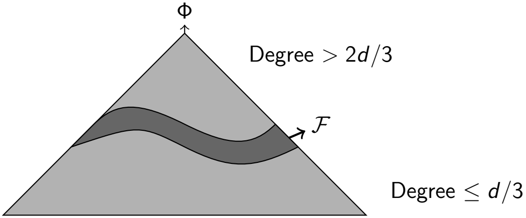

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Interpolate!

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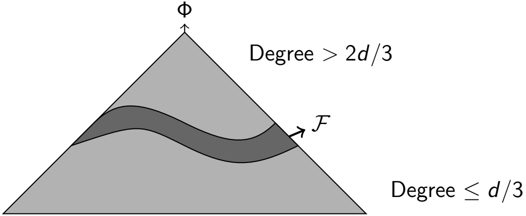

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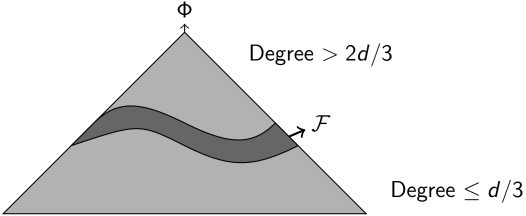

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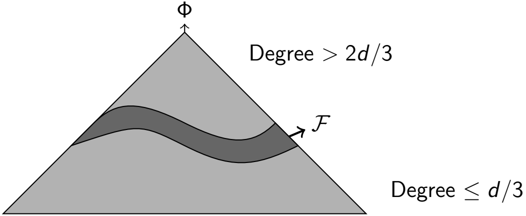

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$$\mathrm{Size}(s, d) = ?$$

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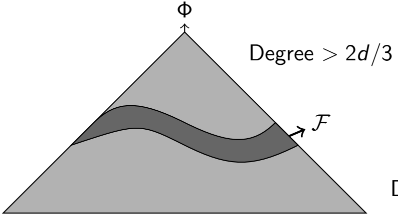

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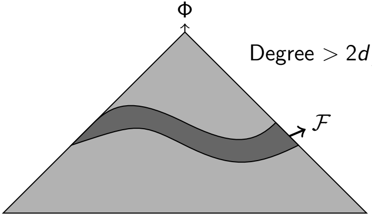
$$\mathrm{Depth}(d) = \qquad O(\log d)$$

$$\mathrm{Size}(s, d) = \qquad s^{O(\log d)}$$

Adapting to circuits: [Hyafil]



$$\mathcal{F} = \left\{ v \in \Phi \mid \frac{d}{3} < \deg(v) \leq \frac{2d}{3} \right\}$$

$$\Phi = \sum_{v_i \in \mathcal{F}} A_i \Phi_{v_i} + \sum_{v_i, v_j \in \mathcal{F}} A_{i,j} \Phi_{v_i} \Phi_{v_j}$$

$$\mathrm{Depth}(d) = O(\log d)$$

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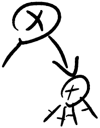

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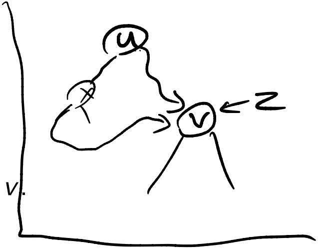

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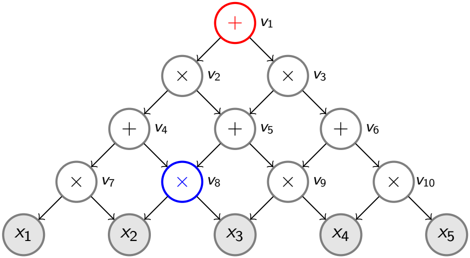
$$[u : v] = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{o/w if } u \text{ is a leaf} \\ [u_1 : v] + [u_2 : v] & \text{if } u = u_1 + u_2 \\ [u_1] \cdot [u_2 : v] & \text{if } u = u_1 \times u_2 \end{cases}$$

An example



$$[v_1 : v_8] \quad =$$

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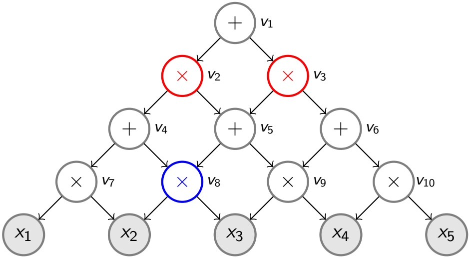

$$[v_1 : v_8] = [v_2 : v_8] + \underbrace{[v_3 : v_8]}$$

$$[v_5]\cdot\underbrace{[v_6 : v_8]}$$

$$[v_9 : v_8] + [v_{10} : v_8]$$

$$[x_3]\cdot\underbrace{[x_4 : v_8]}_{=0} + [x_4]\underbrace{[x_5 : v_8]}_{=0}$$

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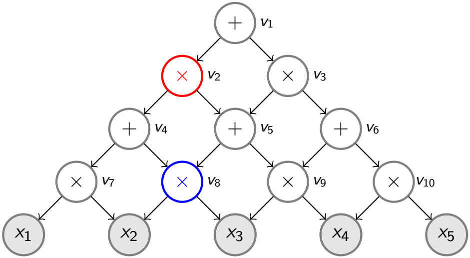

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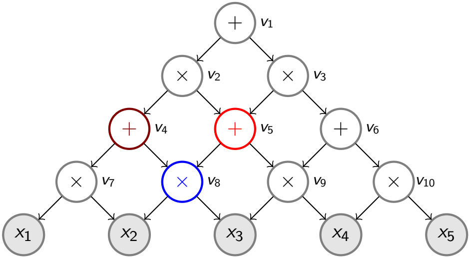

$$\begin{aligned}[v_1 : v_8] &= [v_2 : v_8] \quad + \quad \cancel{[v_3 : v_8]} \\ &= [v_4] \cdot [v_5 : v_8]\end{aligned}$$

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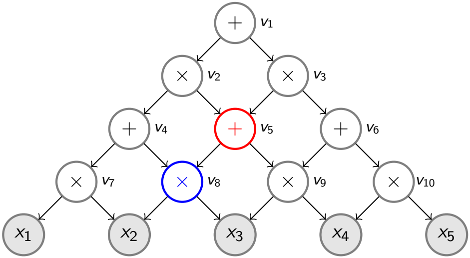

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[v_1 : v_8] &= [v_2 : v_8] \quad + \quad \cancel{[v_3 : v_8]} \\
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&= (x_1 x_2 + x_2 x_3) \cdot [v_5 : v_8]
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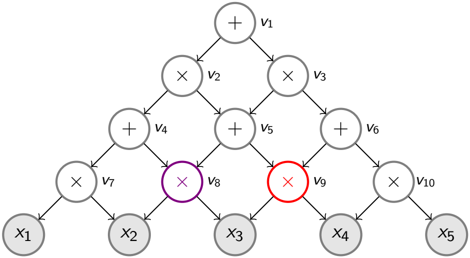

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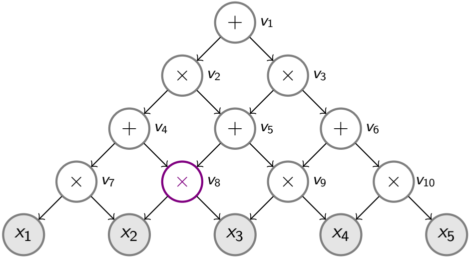

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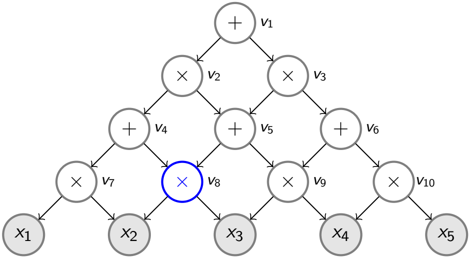

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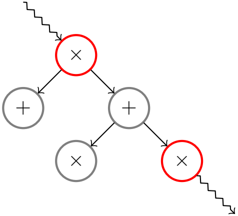

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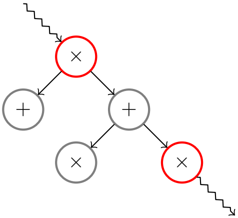

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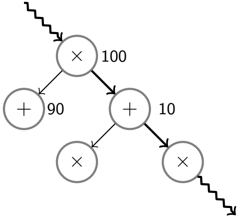

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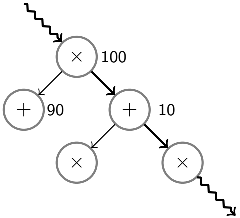

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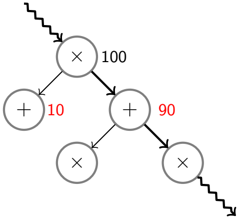

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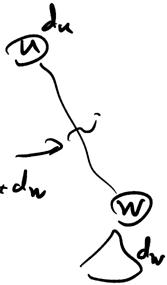

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$$d_u - d_v \leq d_u - \frac{d_u + d_w}{2} = \frac{d_u}{2} - \frac{d_w}{2}$$

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Summarizing

$$[u] = \sum_{v \in \mathcal{F}_a} [u : v] \cdot [v_L] \cdot [v_R]$$

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Theorem ([Valiant-Skyum-Berkowitz-Rackoff])

If Φ is a size s circuit computing an n-variate degree d polynomial f, then there is a circuit Φ' computing f with the following properties.

- *Every gate of Φ' computes either $[u]$, $[u : v]$, or on of the above products, (so size $O(s^4)$)*
- *All addition gates have fan-in at most s^2,*
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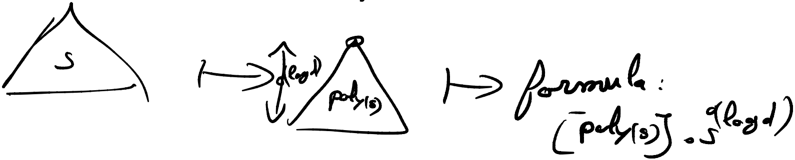
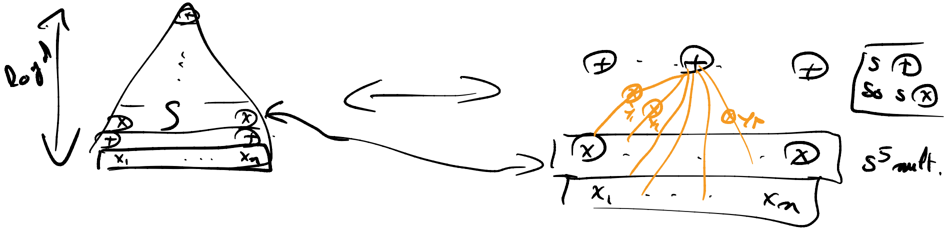
- A sized-s circuit can be simulated by a formula of size $s^{O(\log d)}$.



First consequences of [VSBR]

- A sized-s circuit can be simulated by a formula of size $s^{O(\log d)}$.
- Easy way to construct universal circuits.



Reducing to depth four

Can we reduce the depth further?

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Theorem (Koiran)

If f is computed by a circuit of size s,
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Lemma

If f is computed by an ABP of size s,
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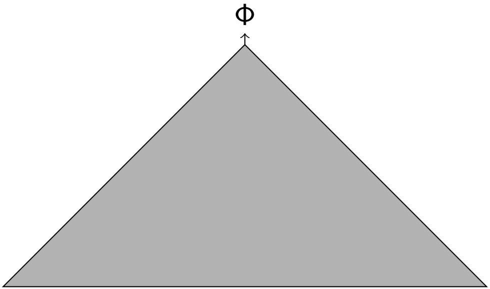
Reducing to depth four : starting from circuits



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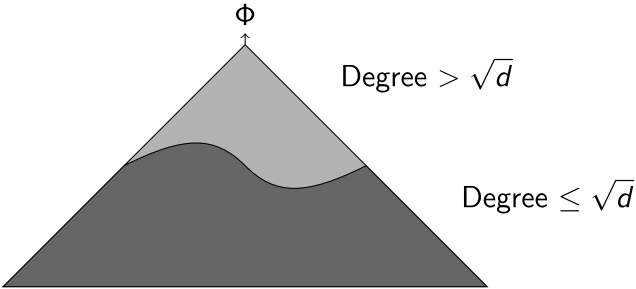

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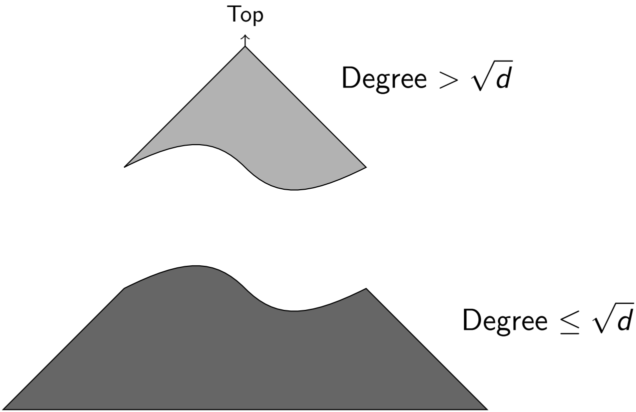

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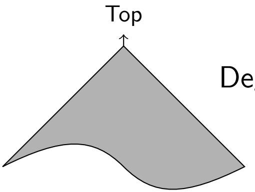
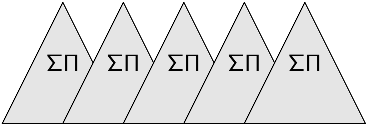


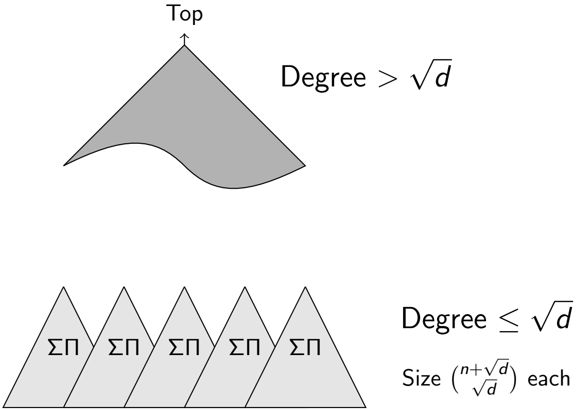
Degree $> \sqrt{d}$



Degree $\leq \sqrt{d}$

Size $\binom{n+\sqrt{d}}{\sqrt{d}}$ each

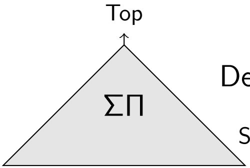
Reducing to depth four : starting from circuits



Lemma ([T.])

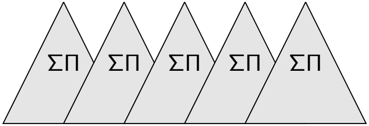
If the circuit has [VSBR] properties,
then $\deg(\mathrm{Top}(z_1, \ldots, z_s)) \leq 15\sqrt{d}$

Reducing to depth four : starting from circuits



Degree $> \sqrt{d}$

Size $\binom{s+15\sqrt{d}}{15\sqrt{d}}$



Degree $\leq \sqrt{d}$

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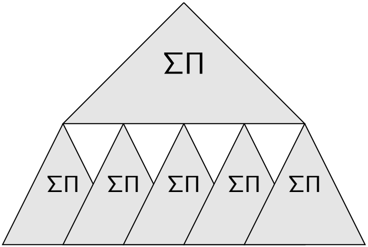
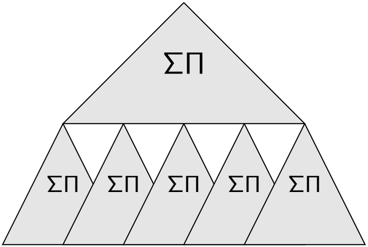


Theorem

Equivalent depth-4 circuit of size

$$s\binom{n+\sqrt{d}}{n} \quad + \quad \binom{s+15\sqrt{d}}{s} \quad = \quad s^{O(\sqrt{d})}$$

Reducing to depth four : starting from circuits



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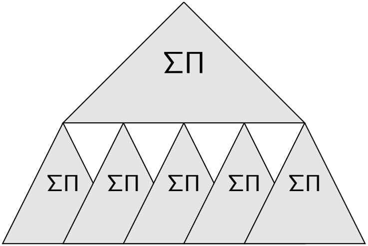
$\square$

Reducing to depth four : starting from circuits



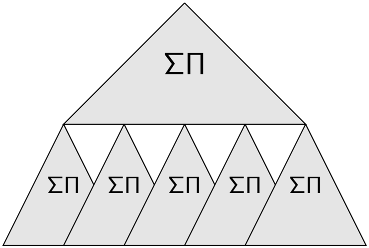
Theorem

Equivalent homogeneous depth-4 circuit with bottom fan-in at most $\sqrt{d}$ of size

$$s\binom{n+\sqrt{d}}{n} \quad + \quad \binom{s+15\sqrt{d}}{s} \quad = \quad s^{O(\sqrt{d})}$$

□

Reducing to depth four : starting from circuits



Theorem

Equivalent homogeneous $\Sigma\Pi\Sigma\Pi^{[\sqrt{d}]}$ *circuit of size*

$$s\binom{n+\sqrt{d}}{n} + \binom{s+15\sqrt{d}}{s} = s^{O(\sqrt{d})}$$

□

[SV]'s proof

Let's start with [VSBR]

$$f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}$$

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This is a $\Sigma\Pi\Sigma\Pi^{[d/2]}$ circuit. We want to obtain a $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit.

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Each f_{ij} is also some $[u : v]$. Keep expanding terms of degree more than t.

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[SV]'s proof

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$$f = \sum_{i=1}^{s} \left(\sum_{j=1}^{s} g_{j1} \cdots g_{j5} \right) \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}$$

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[SV]'s proof

Let's start with [VSBR]

$$f \quad = \quad \sum_{i=1}^{s^2} f_{i1} \cdots f_{i9}$$

This is a $\Sigma\Pi\Sigma\Pi^{[d/2]}$ circuit. We want to obtain a $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit. Each f_{ij} is also some $[u : v]$. Keep expanding terms of degree more than t.

[SV]'s proof

Let's start with [VSBR]

$$f = \sum_{i=1}^{s^3} f_{i1} \cdots f_{i13}$$

This is a $\Sigma\Pi\Sigma\Pi^{[d/2]}$ circuit. We want to obtain a $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit. Each f_{ij} is also some $[u:v]$. Keep expanding terms of degree more than t.

[SV]'s proof

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$$f = \sum_{i=1}^{s^4} f_{i1} \cdots f_{i17}$$

This is a $\Sigma\Pi\Sigma\Pi^{[d/2]}$ circuit. We want to obtain a $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit.
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[SV]'s proof

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Each f_{ij} is also some $[u : v]$. Keep expanding terms of degree more than t.

How many iterations until all degrees are at most t?

Number of iterations

$$g = \sum_{j=1}^{s} g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5}$$

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Observation

In each summand, at least two terms have degree at least $t/8$.

Number of iterations

$$g \quad = \quad \sum_{j=1}^{s} \underbrace{g_{j1}}_{\geq t/5} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5}$$

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In each summand, at least two terms have degree at least $t/8$.

How many factors of degree at least $t/8$?

$$f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}$$

Number of iterations

$$g \quad = \quad \sum_{j=1}^{s} \underbrace{g_{j1}}_{\geq t/5} \cdot \underbrace{g_{j2}}_{\geq t/8} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5}$$

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In each summand, at least two terms have degree at least $t/8$.

How many factors of degree at least $t/8$?

$$f \quad = \quad \sum_{i=1}^{s^2} f_{i1} \cdot f_{i12} \cdot f_{i3} \cdot f_{i4} \cdots f_{i9}$$

Number of iterations

$$g \quad = \quad \sum_{j=1}^{s} \underbrace{g_{j1}}_{\geq t/5} \cdot \underbrace{g_{j2}}_{\geq t/8} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5}$$

Observation

In each summand, at least two terms have degree at least $t/8$.

How many factors of degree at least $t/8$? At most $8d/t$.

$$f \quad = \quad \sum_{i=1}^{s^2} f_{i1} \cdot f_{i12} \cdot f_{i3} \cdot f_{i4} \cdots f_{i9}$$

Number of iterations

$$g \quad = \quad \sum_{j=1}^{s} \underbrace{g_{j1}}_{\geq t/5} \cdot \underbrace{g_{j2}}_{\geq t/8} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5}$$

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$$f \quad = \quad \sum_{i=1}^{s^2} f_{i1} \cdot f_{i12} \cdot f_{i3} \cdot f_{i4} \cdots f_{i9}$$

$$\sum^{s^{O(\sqrt{d})}} \prod^{O(\sqrt{d})} \sum^{s^{O(\sqrt{d})}} \prod^{\sqrt{d}}$$

Final $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit has top fan-in at most $s^{O(d/t)}$.

A better starting point?

Recall

If f has a sized-s circuit, then it has a $\Sigma\Pi\Sigma\Pi^{[\sqrt{d}]}$ of size $s^{O(\sqrt{d})}$.

$$f \quad = \quad \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}$$

If we start with a homogeneous formula, can we do better?

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[Hrubes-Yehudayoff]: Yes!

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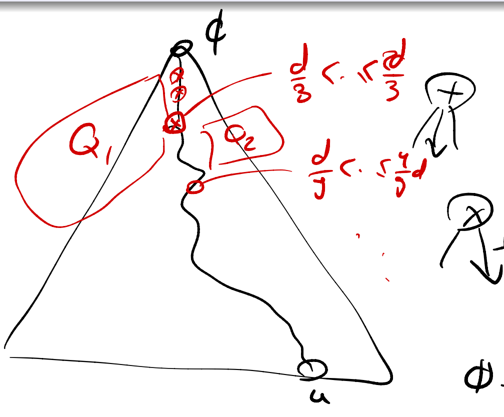
If we start with a homogeneous formula, can we do better?
[Hrubes-Yehudayoff]: Yes!

Lemma ([Hrubes-Yehudayoff])

$$f \quad = \quad \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdots f_{i\ell} \qquad \textit{with } \left(\frac{1}{3}\right)^j \cdot d < \deg(f_{ij}) \leq \left(\frac{2}{3}\right)^j \cdot d$$

Lemma ([Hrubes-Yehudayoff])

$$f \quad = \quad \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdots f_{i\ell} \qquad \textit{with} \left(\frac{1}{3}\right)^{j} \cdot d < \deg(f_{ij}) \leq \left(\frac{2}{3}\right)^{j} \cdot d$$



larger degree

$$\Phi = \Phi_{u \leftarrow 0} + Q_1 \cdot Q_2 \cdots Q_p \cdot u$$

A better starting point?

Recall

If f has a sized-s circuit, then it has a $\Sigma\Pi\Sigma\Pi^{[\sqrt{d}]}$ of size $s^{O(\sqrt{d})}$.

Theorem (Saptharishi?)

If f has a homogeneous sized-s formula,
then it has a homogeneous $\Sigma\Pi^{[\Omega(d \log t/t)]}\Sigma\Pi^{[\sqrt{t}]}$ t.

A better starting point?

Recall

If f has a sized-s circuit, then it has a $\Sigma\Pi\Sigma\Pi^{[\sqrt{d}]}$ of size $s^{O(\sqrt{d})}$.

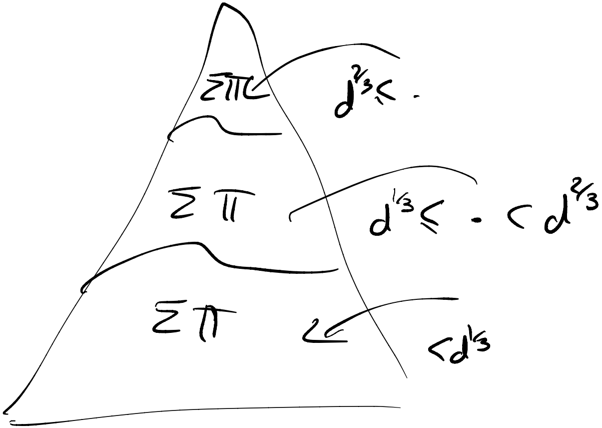
Theorem (Saptharishi?)

If f has a homogeneous sized-s formula, then it has a homogeneous $\Sigma\Pi^{[\Omega(d \log t/t)]}\Sigma\Pi^{[\sqrt{t}]}$.

Theorem (KOS)

If f has a syntactically multilinear sized-s circuit, then it has a $\Sigma\Pi\Sigma\Pi$ of size $2^{O(\sqrt{N \log s})}$.

Generalization to homogeneous depth-2Δ



$\Sigma\Pi\Sigma\Pi\Sigma\Pi$

$s^{O(d^{1/3})}$

Generalization to homogeneous depth-2Δ

Theorem

If f has a sized-s circuit, then it has a depth-2Δ $\underbrace{\Sigma\Pi^{[O(d^{1/\Delta})]}\Sigma\Pi^{[O(d^{1/\Delta})]}\dots\Sigma\Pi^{[O(d^{1/\Delta})]}}_{\left(\Sigma\Pi^{[O(d^{1/\Delta})]}\right)^{\Delta}}$ *of size* $s^{O(\Delta\cdot d^{1/\Delta})}$.

Generalization to homogeneous depth-2Δ

Theorem

If f has a sized-s circuit,
then it has a depth-2Δ $\Sigma\Pi^{[O(d^{1/\Delta})]}\Sigma\Pi^{[O(d^{1/\Delta})]}\dots\Sigma\Pi^{[O(d^{1/\Delta})]}$ of size $s^{O(\Delta\cdot d^{1/\Delta})}$.

Theorem

If f has a sized-$\mathrm{poly}(N)$ syntactically multilinear circuit,
then it has a $(\Sigma\Pi)^{\Delta}$ of size $s^{O\left(\Delta\cdot(n/\log s)^{1/\Delta}\right)}$.

Reduction to Depth-3 Circuits

(or, "can we do better if we allow the final circuit to be highly inhomogeneous?")

Road map [GKKS]

$$\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$$
circuits

$$\downarrow$$

$$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$$
circuits

$$\downarrow$$

$$\sum \prod \sum$$
circuits

Road map [GKKS]

$$\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$$
circuits

$$\Big\downarrow \text{ App. of Ryser's formula}$$

$$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$$
circuits

$$\Big\downarrow$$

$$\sum \prod \sum$$
circuits

Road map [GKKS]

$$\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$$

circuits

$$\Big\downarrow \text{App. of Ryser's formula}$$

$$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum$$

circuits

$$\Big\downarrow \text{[Saxena]'s duality trick}$$

$$\sum \prod \sum$$

circuits

Road map [GKKS]

$$\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}} \text{ circuits}$$

Only over $\mathbb{Q}$,$\mathbb{R}$ etc. $\Big\downarrow$ App. of Ryser's formula

$$\sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum \text{ circuits}$$

Heavily non-homogeneous $\Big\downarrow$ [Saxena]'s duality trick

$$\sum \prod \sum \text{ circuits}$$

$$\prod^{d} \Big\downarrow \sum^{2^d} \bigwedge^{d} \sum^{d}$$

$$\bigwedge^{a} \sum^{m} \bigwedge^{b} \Big\downarrow \sum^{\mathrm{poly}(a,m,b)} \prod^{[a\,m\,b]} \sum^{[2]}$$

Step 1: $\Pi^{[d]}$ to $\Sigma^{[2^d]}\wedge^{[d]}\Sigma^{[d]}$

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Recall Ryser's formula:

$$\mathrm{Perm}_d \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & \ddots & \vdots \\ x_{d1} & \cdots & x_{dd} \end{bmatrix} = \sum_{S \subseteq [d]} (-1)^{d-|S|} \prod_{i=1}^{d} \sum_{j \in S} x_{ij}$$

Step 1: $\Pi^{[d]}$ to $\Sigma^{[2^d]}\wedge^{[d]}\Sigma^{[d]}$

Recall Ryser's formula:

$$\mathrm{Perm}_d \begin{bmatrix} x_1 & \cdots & x_d \\ \vdots & \ddots & \vdots \\ x_1 & \cdots & x_d \end{bmatrix} = \sum_{S\subseteq[d]} (-1)^{d-|S|} \prod_{i=1}^{d} \sum_{j\in S} x_j$$

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Step 1: $\Pi^{[d]}$ to $\Sigma^{[2^d]}\wedge^{[d]}\Sigma^{[d]}$

Recall Ryser's formula:

$$d! \cdot x_1 \ldots x_d \quad = \quad \sum_{S \subseteq [d]} (-1)^{d-|S|} \left(\sum_{j \in S} x_j \right)^d$$

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[Fischer]:

$$d! \cdot x_1 \dots x_d \quad = \quad \sum_{S \subseteq [d]} (-1)^{d-|S|} \left(\sum_{j \in S} x_j \right)^d$$

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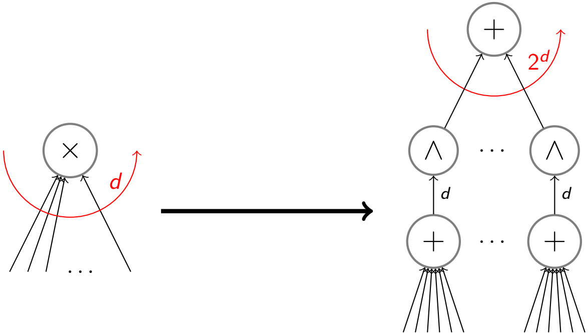

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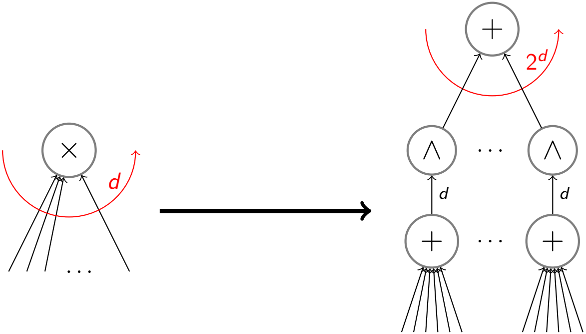
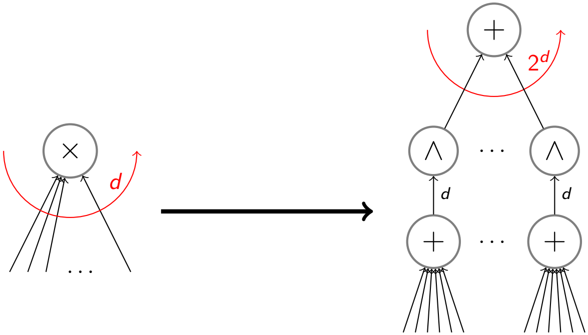


$$\prod^{d} \longrightarrow \sum^{2^d}\bigwedge^{d}\sum^{d}$$

Step 1: $\Pi^{[d]}$ to $\Sigma^{[2^d]}\wedge^{[d]}\Sigma^{[d]}$



$$\prod^{d} \longrightarrow \sum^{2^d}\bigwedge^{d}\sum^{d}$$

$$\sum\underbrace{\prod^{\sqrt{d}}}_{\sum^{2^{\sqrt{d}}}\bigwedge^{\sqrt{d}}\sum^{\sqrt{d}}}\sum\underbrace{\prod^{\sqrt{d}}}_{\sum^{2^{\sqrt{d}}}\bigwedge^{\sqrt{d}}\sum^{\sqrt{d}}} \text{ of size } s \longrightarrow \sum\bigwedge^{\sqrt{d}}\sum\bigwedge^{\sqrt{d}}\sum \text{ of size } 2^{O(\sqrt{d})}\cdot s$$

Road map

$$\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$$

circuits

$$\downarrow$$

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circuits

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circuits

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circuits

Step 2: $\wedge^{[a]}\Sigma^{[s]}\wedge^{[b]}$ to $\Sigma^{[\mathrm{poly}(s,a,b)]}\Pi^{[sbd]}\Sigma^{[2]}$

Step 2: $\wedge^{[a]}\sum^{[s]}\wedge^{[b]}$ to $\sum^{[\mathrm{poly}(s,a,b)]}\prod^{[sbd]}\sum^{[2]}$

$$T = \left(x_1^b + \cdots + x_s^b\right)^a$$

Step 2: $\wedge^{[a]}\Sigma^{[s]}\wedge^{[b]}$ to $\Sigma^{[\mathrm{poly}(s,a,b)]}\Pi^{[sbd]}\Sigma^{[2]}$

$$T \quad = \quad \left(x_1^b + \cdots + x_s^b\right)^a$$

Lemma ([Saxena])

There exists univariate polynomials f_{ij}'s of degree at most a such that

$$\ell^a = (y_1 + \cdots + y_s)^a \quad = \quad \sum_{i=1}^{O(sa^2)} \prod_{j=1}^{s} f_{ij}(x_j)$$

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Sketch of a proof by Gupta-Forbes-Shpilka

$$P_{\mathbf{y}}(t) \;=\; (1 + y_1 t)\ldots(1 + y_s t) = \;\; 1 + \ell t + (\text{higher degree terms}) \to s$$

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Sketch of a proof by Gupta-Forbes-Shpilka

$$P_{\mathbf{y}}(t) - 1 = \qquad \ell t \quad + \quad \text{(higher degree terms)} \to s$$

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$$\ell^a = (y_1 + \cdots + y_s)^a \quad = \quad \sum_{i=1}^{O(sa^2)} \prod_{j=1}^{s} f_{ij}(x_j)$$

Sketch of a proof by Gupta-Forbes-Shpilka

$$(P_{\mathbf{y}}(t) - 1)^a = \qquad \ell^a t^a \quad + \quad \text{(higher degree terms)} \to sa$$

Step 2: $\wedge^{[a]}\Sigma^{[s]}\wedge^{[b]}$ to $\Sigma^{[\mathrm{poly}(s,a,b)]}\Pi^{[sbd]}\Sigma^{[2]}$

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Interpolate!

$P_y(t)^\ell$ $(0 \le \ell \le a+1)$

$(P_{\mathbf{y}}(t) - 1)^a$ expanded is a sum of $(a+1)$ product of univariates. □

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$$(y_1 + \cdots + y_s)^a = \sum_i^{\mathrm{poly}(s,a)} \prod_{j=1}^{s} f_{ij}(y_j)$$

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where $\tilde{f}_{ij}(t) \; := \; f_{ij}(t^{\sqrt{d}})$

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Note that $\tilde{f}_{ij}(t)$ is a univariate polynomial that can be factorized over $\mathbb{C}$:

$$\tilde{f}_{ij}(t) = \prod_{k=1}^{ab}(t - \zeta_{ijk})$$

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... a $\Sigma\Pi\Sigma$ circuit of $\mathrm{poly}(s, a, b)$ size.

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Putting it together

general circuit
of size s

Putting it together

general circuit of size s $\longrightarrow$ $\sum\prod^{\sqrt{d}}\sum\prod^{\sqrt{d}}$ circuit of size $s^{O(\sqrt{d})}$

Putting it together

$$
\begin{tikzcd}
\begin{array}{c}\text{general circuit}\\ \text{of size } s\end{array} \arrow[r] & \begin{array}{c}\sum\prod^{\sqrt{d}}\sum\prod^{\sqrt{d}} \text{ circuit}\\ \text{of size } s^{O(\sqrt{d})}\end{array} \arrow[d] \\
 & \begin{array}{c}\sum\bigwedge^{\sqrt{d}}\sum\bigwedge^{\sqrt{d}}\sum \text{ circuits}\\ \text{of size } s^{O(\sqrt{d})}\cdot 2^{O(\sqrt{d})}\end{array}
\end{tikzcd}
$$

Putting it together

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Question: Where should one try to prove lower bounds?

Putting it together

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\sum \prod \sum \text{ circuits of size } s^{O(\sqrt{d})} & \sum \bigwedge^{\sqrt{d}} \sum \bigwedge^{\sqrt{d}} \sum \text{ circuits of size } s^{O(\sqrt{d})} \arrow[l]
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Other constants for the depth?

Recall

If f has a sized-s circuit,
then it has a depth-2Δ $\left(\Sigma\Pi^{[O(d^{1/\Delta})]}\right)^{\Delta}$ of size $s^{O(\Delta \cdot d^{1/\Delta})}$.

$\Sigma\Pi^{d^{1/3}}\Sigma\Pi^{d^{1/3}}\Sigma\Pi^{d^{1/3}}$ $\quad s^{O(d^{1/3})}$

$\Sigma\wedge^{d^{1/3}}\Sigma \quad \Sigma\wedge^{d^{1/3}}\Sigma$

$\Sigma\Pi\Sigma$

$\Sigma\Pi^{s^{O(d^{1/3})}}\Sigma\Pi^{d^{1/3}} \quad \mapsto s^{O(d^{1/3})}$

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Theorem

If f has a sized-s circuit,
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Corollary

- Det_n *has a* $\Sigma\Pi\Sigma\Pi$ *of size* $n^{O(\sqrt[3]{n})}$.
- $\mathrm{IMM}_{n,d}$ *has a* $\Sigma\Pi\Sigma\Pi$ *of size* $n^{O(\sqrt[3]{d})}$.
- *If* Perm_n *needs* $\Sigma\Pi\Sigma\Pi$ *of size* $n^{\omega(\sqrt[3]{n})}$, *then* $VP \neq VNP$.

Back to the homogeneization (case of constant depth)

- All gates compute *homogeneous polynomials*.
- Hence, no gate can compute polynomials of degree larger than output.
- For circuits and ABPs, homogeneity can be assumed without loss of generality.
  For formulas, probably not.
  For constant depth formulas, certainly not.

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What happens if we allow some subexponential blow up?

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Theorem (Raz)

If f computed by a formula of size s,
then it is computed by a homogeneous one of size $2^{O(d \log \log s)}$.

Back to the homogeneization (case of constant depth)

Theorem (GKKS)

If f computed by a circuit of size s and depth 3, then it is computed by a homogeneous one of size $\mathrm{poly}(s)2^{O(\sqrt{d})}$ and depth 5.

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If f computed by a circuit of size s and depth 3, then it is computed by a homogeneous one of size $\mathrm{poly}(s)2^{O(\sqrt{d})}$ and depth 5.

Theorem (LST)

If f computed by a circuit of size s and depth Γ, then it is computed by a homogeneous one of size $\mathrm{poly}(s)2^{O(\sqrt{d})}$ and depth 2Γ-1.

Thank you.