Algebraic Complexity: Structural results
Depth reduction, Homogeneization, Multilinearization, …

Sébastien Tavenas

October 15th, 2021
Overview

1. Why to look at structural results

2. Homogeneization / (Set)-multilinearization
   - Homogeneization
   - Multilinearization

3. Parallelization
   - Classical depth reductions of [Brent] and [VSBR]
   - To constant depth
General roadmap for lower bounds
Four steps in most lower bound proofs

Step 1: Finding a “nice” form for the model
Four steps in most lower bound proofs

Step 1: Finding a “nice” form for the model

Meta Theorem 1

Every small circuit can be equivalently computed as a “nice” triangle.
Four steps in most lower bound proofs

Step 1: Finding a “nice” form for the model

Meta Theorem 1

Every small circuit can be equivalently computed as a “nice” form.
Four steps in most lower bound proofs

Step 1: Finding a “nice” form for the model

Meta Theorem 1
Every small circuit can be equivalently computed as a “nice” △.

Step 2: Constructing a complexity measure

Meta Theorem 2
Find a map $\Gamma : \mathbb{F}[x] \to \mathbb{Z}_{\geq 0}$ such that $\Gamma(\triangle)$ is small.
Four steps in most lower bound proofs

Step 1: Finding a “nice” form for the model

Meta Theorem 1
Every small circuit can be equivalently computed as a “nice” ▲

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Meta Theorem 2
Find a map $\Gamma : F[x] \rightarrow \mathbb{Z}_{\geq 0}$ such that $\Gamma(▲)$ is small.

Step 3: Heuristic estimate for a random polynomial

Meta Theorem 2
Convince yourself that $\Gamma(R)$ must be LARGE for a random polynomial $R$. 
Four steps in most lower bound proofs

**Step 1: Finding a “nice” form for the model**

*Meta Theorem 1*

Every small circuit can be equivalently computed as a “nice” form.

**Step 2: Constructing a complexity measure**

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Find a map $\Gamma : \mathbb{F}[x] \rightarrow \mathbb{Z}_{\geq 0}$ such that $\Gamma (\triangle)$ is small.

**Step 3: Heuristic estimate for a random polynomial**

*Meta Theorem 2*

Convince yourself that $\Gamma (R)$ must be LARGE for a random polynomial $R$.

**Step 4: Find a hay in the haystack**

Structural results
Four steps in most lower bound proofs

Step 1: Finding a “nice” form for the model

Meta Theorem 1
Every small circuit can be equivalently computed as a “nice”
Four steps in most lower bound proofs

Step 1: Finding a “nice” form for the model

Meta Theorem 1

Every small circuit can be equivalently computed as a “nice” form.

Homogenization, (Set)-multilinearization, Depth reduction
Arithmetic models

Formulas $\subseteq$ ABP $\subseteq$ Circuits
Arithmetic models

Formulas $\subseteq$ ABP $\subseteq$ Circuits

- Reverse inclusions?
Arithmetic models

Formulas $\subseteq$ ABP $\subseteq$ Circuits

- Reverse inclusions?
- Circuit of size $s \leadsto$
Arithmetic models

Formulas $\subseteq$ ABP $\subseteq$ Circuits

- Reverse inclusions?
- Circuit of size $s \rightsquigarrow$ Formula of size $s^{O(\log d)}$. 
Few words about fan-ins

If nothing is mentioned:
- For circuits, formula of “large depth”:
  - $+$-gate: unbounded
  - $\star$-gate: constant
- For circuits, formula of constant depth:
  - $+$-gate: unbounded
  - $\star$-gate: unbounded
Homogenization (basics)

- All gates compute *homogeneous polynomials*.
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- Hence, *no gate can compute polynomials of degree larger than output*.
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- Hence, **no gate can compute polynomials of degree larger than output**.
- For circuits and ABPs, homogeneity can be assumed without loss of generality.
  For formulas, probably not.
  For constant depth formulas, certainly not.

\[
\begin{align*}
\text{(Det}_n) & \in \Sigma \Pi \Sigma \Pi \\
\text{non \ hom} & \rightarrow n^{O(\sqrt{n})} \\
\text{hom} & \rightarrow 2^{O(\sqrt{n})}
\end{align*}
\]
Homogenization (basics)

- All gates compute *homogeneous polynomials*.
- Hence, no gate can compute polynomials of degree larger than output.
- For circuits and ABPs, homogeneity can be assumed without loss of generality.
  For formulas, probably not.
  For constant depth formulas, certainly not.

\[
g = g_1 	imes g_2 \rightarrow g^{(i)} = \sum_{j=0}^{i} g_1^{(j)} \times g_2^{(i-j)}
\]

\[
g = \sum_{j} g_j \rightarrow g^{(i)} = \sum_{j} g_j^{(i)}
\]

\[
\text{Structural results}
\]

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(Syntactic) (Set)-multilinearization

- Multilinear, Set-multilinear
(Syntactic) (Set)-multilinearization

- Multilinear, Set-multilinear
- Semantic vs. Syntactic

A circuit syntactic set-multilinear

\[ * (\emptyset) \to \exists \exists \exists \]
\[ * \exists x \to \exists \exists x \exists x \]
\[ * (x \cdot y) \to s_1 = s_2 = \ldots = s_p = s \]
\[ * (x_1 \cdot x_2) \to s_1 = s_1 + s_2 \]

A circuit is called synt multilinear

\[ * (\emptyset) \to \exists \exists \exists \]
\[ * \exists x \to \exists \exists x \exists x \]
\[ * (x \cdot y) \to s \cup s \]
\[ * (x \cdot y) \to s \cup s \]
**Multilinear, Set-multilinear**

**Semantic vs. Syntactic**

**Expensive!**

<table>
<thead>
<tr>
<th></th>
<th>Syn. Multilinear</th>
<th>Syn. Set-multilinear</th>
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<tbody>
<tr>
<td>Circuits</td>
<td>???</td>
<td>$s \cdot 2^{O(d)}$</td>
</tr>
<tr>
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Other depth reductions in lower bounds

Multilinear formulas

\[ f = \sum_{i=1}^{s} g_{i1} \cdot g_{i2} \cdots g_{i\ell}, \quad \left(\frac{1}{3}\right)^j \cdot n \leq \text{Var}(g_{ij}) \leq \left(\frac{2}{3}\right)^j \cdot n \]

[\text{Raz, Raz-Yehudayoff}]
Other depth reductions in lower bounds

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<td>$f = \sum_{i=1}^{s} g_{i1} \cdot g_{i2} \cdots g_{i\ell}$, $(1/3)^j \cdot d \leq \text{deg}(g_{ij}) \leq (2/3)^j \cdot d$</td>
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Depth reducing formulas

\[ \phi \]
Depth reducing formulas

\[ \frac{s}{3} \leq |\text{subtree}| \leq \frac{2s}{3} \]
Depth reducing formulas

\[ \Phi_1 \]

\[ \Phi_2 \]
Depth reducing formulas

\[ \Phi_1(z) = A \cdot z + B \]

\[ \Phi = A \cdot \Phi_2 + B \]
Depth reducing formulas

\[ \Phi_1(z) = A \cdot z + B \]
\[ \Phi = A \cdot \Phi_2 + B = (\Phi_1(1) - \Phi_1(0)) \cdot \Phi_2 + \Phi_1(0) \]
Depth reducing formulas

\[
\Phi_1(z) = A \cdot z + B \\
\Phi = A \cdot \Phi_2 + B = (\Phi_1(1) - \Phi_1(0)) \cdot \Phi_2 + \Phi_1(0)
\]
Depth reducing formulas

\[
\begin{align*}
\text{Size}(s) & \leq 4 \cdot \text{Size}(2s/3) + O(1) \\
\text{Depth}(s) & \leq \text{Depth}(2s/3) + O(1)
\end{align*}
\]
Depth reducing formulas

\[
\begin{align*}
\text{Size}(s) & \leq 4 \cdot \text{Size}(2s/3) + O(1) \implies \text{poly}(s) \\
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\]
Adapting to circuits
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\[ \frac{s}{3} \leq \text{size} \leq \frac{2s}{3} \]
Adapting to circuits

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Not true for circuits!
Adapting to circuits

\[ \frac{d}{3} \leq \text{degree} \leq \frac{2d}{3} \] (start with a homogeneous circuit)
Adapting to circuits

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Multiple paths from root!
Adapting to circuits: Attempt 1
Adapting to circuits: Attempt 1

\[
\mathcal{F} = \left\{ v \in \Phi \mid \frac{d}{3} < \deg(v) \leq \frac{2d}{3} \right\}
\]
Adapting to circuits: Attempt 1

\[ F = \left\{ v \in \Phi \mid \frac{d}{3} < \text{deg}(v) \leq \frac{2d}{3} \right\} \]

\[ \Phi = \sum_{v_i \in F} A_i \Phi_{v_i} + \sum_{v_i, v_j \in F} A_i,j \Phi_{v_i} \Phi_{v_j} \]
Adapting to circuits: Attempt 1

\[ \mathcal{F} = \left\{ \nu \in \Phi \mid \frac{d}{3} < \deg(\nu) \leq \frac{2d}{3} \right\} \]

\[ \Phi = \sum_{\nu_i \in \mathcal{F}} A_i \Phi_{\nu_i} + \sum_{\nu_i, \nu_j \in \mathcal{F}} A_{i,j} \Phi_{\nu_i} \Phi_{\nu_j} \]

each have degree at most \(2d/3\)
Adapting to circuits: Attempt 1

\[ \mathcal{F} = \left\{ v \in \Phi \mid \frac{d}{3} < \deg(v) \leq \frac{2d}{3} \right\} \]

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each have degree at most \( 2d/3 \)

Interpolate!
Adapting to circuits: Attempt 1

\[ \Phi \]

Degree \( > 2d/3 \)

Degree \( \leq d/3 \)

\[ \mathcal{F} = \left\{ v \in \Phi \mid \frac{d}{3} < \text{deg}(v) \leq \frac{2d}{3} \right\} \]

\[ \Phi = \sum_{v_i \in \mathcal{F}} A_i \Phi_{v_i} + \sum_{v_i, v_j \in \mathcal{F}} A_{i,j} \Phi_{v_i} \Phi_{v_j} \]

\[ \text{Depth}(d) = \text{Depth}(2d/3) + O(1) \]
Adapting to circuits: Attempt 1

\begin{align*}
\mathcal{F} &= \left\{ v \in \Phi \mid \frac{d}{3} < \deg(v) \leq \frac{2d}{3} \right\} \\
\Phi &= \sum_{v_i \in \mathcal{F}} A_i \Phi_{v_i} + \sum_{v_i, v_j \in \mathcal{F}} A_{i,j} \Phi_{v_i} \Phi_{v_j} \\
\text{Depth}(d) &= O(\log d)
\end{align*}
Adapting to circuits: Attempt 1

\[ \Phi \]

\[ \text{Degree} > \frac{2d}{3} \]

\[ \text{Degree} \leq \frac{d}{3} \]

\[ \mathcal{F} = \left\{ v \in \Phi \mid \frac{d}{3} < \deg(v) \leq \frac{2d}{3} \right\} \]

\[ \Phi = \sum_{v_i \in \mathcal{F}} A_i \Phi_{v_i} + \sum_{v_i, v_j \in \mathcal{F}} A_{i,j} \Phi_{v_i} \Phi_{v_j} \]

\[ \text{Depth}(d) = O(\log d) \]

\[ \text{Size}(s, d) = \]

\[ ? \]
Adapting to circuits: Attempt 1

\[ \mathcal{F} = \left\{ v \in \Phi \mid \frac{d}{3} < \deg(v) \leq \frac{2d}{3} \right\} \]

\[ \Phi = \sum_{v_i \in \mathcal{F}} A_i \Phi_{v_i} + \sum_{v_i, v_j \in \mathcal{F}} A_{i,j} \Phi_{v_i} \Phi_{v_j} \]

Depth\((d) = O(\log d)\)

Size\((s, d) = s^O(\log d)\)
Adapting to circuits: [Hyafil]

\[ \Phi \]

Degree > \( \frac{2d}{3} \)

Degree ≤ \( \frac{d}{3} \)

\[ \mathcal{F} = \left\{ v \in \Phi \mid \frac{d}{3} < \deg(v) \leq \frac{2d}{3} \right\} \]

\[ \Phi = \sum_{v_i \in \mathcal{F}} A_i \Phi_v + \sum_{v_i, v_j \in \mathcal{F}} A_{i,j} \Phi_{v_i} \Phi_{v_j} \]

\[ \text{Depth}(d) = O(\log d) \]

\[ \text{Size}(s, d) = sO(\log d) \]
Adapting to circuits: Attempt 2

- Want an analogue of $\Phi = A \cdot \Phi_v + B$. 
Adapting to circuits: Attempt 2

- Want an analogue of $\Phi = A \cdot \Phi_v + B$.
- Problem is that there are multiple paths to $v$. 

```markdown
\[ V \text{SBR}: \text{Do not look at all paths. Only take a canonical path, like say taking the right-edge out of every } \rightarrow \text{-gate.} \]
\[ u: v = 0 \text{ if } u = v \text{ other wise if } u \text{ is a leaf} \]
\[ u_1: v + u_2: v \text{ if } u = u_1 + u_2 \]
\[ u_1: v \cdot u_2: v \text{ if } u_1 = u_1 \rightarrow u_2 \]
```
Adapting to circuits: Attempt 2

- Want an analogue of $\Phi = A \cdot \Phi_v + B$.
- Problem is that there are multiple paths to $v$. $\Phi$ isn’t really an affine function in $\Phi_v$. 

Adapting to circuits: Attempt 2

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[VSBR]: Do not look at all paths. Only take a canonical path, like say taking the right-edge out of every $\times$-gate.
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$$[u : v] = \begin{cases} 
1 & \text{if } u = v \\
0 & \text{o/w if } u \text{ is a leaf} \\
[u_1 : v] + [u_2 : v] & \text{if } u = u_1 + u_2 \\
[u_1] \cdot [u_2 : v] & \text{if } u = u_1 \times u_2
\end{cases}$$
An example

\[
\begin{bmatrix}
    v_1 \\
    v_2 \\
    v_3 \\
    v_4 \\
    v_5 \\
    v_6 \\
    v_7 \\
    v_8 \\
\end{bmatrix} = \\
\begin{bmatrix}
    v_1 \\
    v_2 \\
    v_3 \\
    v_4 \\
    v_5 \\
    v_6 \\
    v_7 \\
    v_8 \\
\end{bmatrix}
\]
An example

\[ [v_1 : v_8] = [v_2 : v_8] + [v_3 : v_8] \]

\[
= [v_5 : v_8] \cdot [v_6 : v_8] + [v_9 : v_8] + [v_{10} : v_8] + [x_3 : x_4 : x_5 : v_8]
\]

\[ = 0 \]
An example

\[
\begin{align*}
[ v_1 : v_8 ] &= [ v_2 : v_8 ] + [ v_3 : v_8 ]
\end{align*}
\]
An example

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[v_1 : v_8] = [v_2 : v_8] + [v_3 : v_8] \\
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[v_1 : v_8] &= [v_2 : v_8] + [v_3 : v_8] \\
&= [v_4] \cdot [v_5 : v_8] \\
&= (x_1 x_2 + x_2 x_3) \cdot [v_5 : v_8]
\end{align*}
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\]
We want a set of nodes $\mathcal{F}$ such that

$$[u] = \sum_{v \in \mathcal{F}} [u : v] \cdot [v]$$

What are candidates for $\mathcal{F}$?
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$$F_a = \{ v \mid \deg(v) \geq a, \deg(v_L), \deg(v_R) < a \}$$
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Make the circuit *right heavy*.
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**Lemma**

$$[u] = \sum_{v \in \mathcal{F}_a} [u : v] \cdot [v]$$

$$[u : w] = \sum_{v \in \mathcal{F}_a} [u : v] \cdot [v : w]$$
\[ F_a = \{ v \mid \deg(v) \geq a, \ \deg(v_L), \deg(v_R) < a \} \]

\[ [u] = \sum_{v \in F_a[u]} [u : v] \cdot [v] \]
\[ \mathcal{F}_a = \{ v \mid \deg(v) \geq a, \deg(v_L), \deg(v_R) < a \} \]

\[ [u] = \sum_{v \in \mathcal{F}_a[u]} [u : v] \cdot [v_L] \cdot [v_R] \]
\[ \mathcal{F}_a = \{ v \mid \deg(v) \geq a, \deg(v_L), \deg(v_R) < a \} \]

\[ [u] = \sum_{v \in \mathcal{F}_{a[u]}} [u : v] \cdot [v_L] \cdot [v_R] \quad a_{[u]} = \frac{\deg(u)}{2} \]
继续...

\[ \mathcal{F}_a = \{ v \mid \deg(v) \geq a, \deg(v_L), \deg(v_R) < a \} \]

\[ [u] = \sum_{v \in \mathcal{F}_a[u]} [u : v] \cdot [v_L] \cdot [v_R] \quad a_{[u]} = \deg(u)/2 \]
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\[ [u : w] = \sum_{v \in \mathcal{F}_{a[u:w]}} [u : v] \cdot [v : w] \]
\[
\mathcal{F}_a = \{ v \mid \text{deg}(v) \geq a, \text{deg}(v_L), \text{deg}(v_R) < a \}
\]

\[
[u] = \sum_{v \in \mathcal{F}_a[u]} [u : v] \cdot [v_L] \cdot [v_R]
\]

\[
a[u] = \text{deg}(u)/2
\]

\[
\frac{d_u - d_w}{2} + d_w
\]

\[
[u : w] = \sum_{v \in \mathcal{F}_a[u:w]} [u : v] \cdot [v_L] \cdot [v_R : w]
\]
\[ \mathcal{F}_a = \{ v \mid \deg(v) \geq a, \deg(v_L), \deg(v_R) < a \} \]

\[
[u] = \sum_{v \in \mathcal{F}_a[u]} [u : v] \cdot [v_L] \cdot [v_R] \\
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[u : w] = \sum_{v \in \mathcal{F}_a[u:w]} [u : v] \cdot [v_L] \cdot [v_R : w] \\
a_{[u:w]} = \frac{\deg(u) + \deg(w)}{2}
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\[ d_u - d_v \leq d_w - \frac{d_u + d_w}{2} = \frac{d_u}{2} - \frac{d_w}{2} \]
\[ F_a = \{ v \mid \deg(v) \geq a, \deg(v_L), \deg(v_R) < a \} \]

\[
[u] = \sum_{v \in F_a[u]} [u : v] \cdot [v_L] \cdot [v_R] \quad a[u] = \frac{\deg(u)}{2}
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\]

\[
\frac{\frac{\Delta u + \Delta w}{2} - \Delta w}{2} = \frac{\Delta u - \Delta w}{2}
\]
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\[ = \sum_{v \in \mathcal{F}_a[u:w]} [u : v] \cdot \left( \sum_{q \in \mathcal{F}_a[v]} [v_L : q] \cdot [q_L] \cdot [q_R] \right) \cdot [v_R : w] \]
\[ \mathcal{F}_a = \{ v \mid \deg(v) \geq a, \deg(v_L), \deg(v_R) < a \} \]

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[u] = \sum_{v \in \mathcal{F}_a[u]} [u : v] \cdot [v_L] \cdot [v_R] \quad a_{[u]} = \frac{\deg(u)}{2}
\]

\[
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\[ \mathcal{F}_a = \{ \nu \mid \deg(\nu) \geq a, \deg(\nu_L), \deg(\nu_R) < a \} \]

\[ [u] = \sum_{\nu \in \mathcal{F}_a[u]} [u : \nu] \cdot [\nu_L] \cdot [\nu_R] \quad a[u] = \deg(u)/2 \]

\[ [u : w] = \sum_{\nu \in \mathcal{F}_a[u:w]} [u : \nu] \cdot [\nu_L] \cdot [\nu_R : w] \quad a[u:w] = \frac{\deg(u) + \deg(w)}{2} \]

\[ = \sum_{\nu \in \mathcal{F}_a[u:w]} [u : \nu] \cdot \left( \sum_{q \in \mathcal{F}_a[v]} [\nu_L : q] \cdot [q_L] \cdot [q_R] \right) \cdot [\nu_R : w] \]
Summarizing

\[
[u] = \sum_{v \in F_a} [u : v] \cdot [v_L] \cdot [v_R]
\]

\[
[u : w] = \sum_{v \in F_a} \sum_{q \in F_a} [u : v] \cdot [v : q] \cdot [q_L] \cdot [q_R] \cdot [v_R : w]
\]

**Theorem ([Valiant-Skyum-Berkowitz-Rackoff])**

If \( \Phi \) is a size \( s \) circuit computing an \( n \)-variate degree \( d \) polynomial \( f \), then there is a circuit \( \Phi' \) computing \( f \) with the following properties.

- Every gate of \( \Phi' \) computes either \([u], [u : v], \) or one of the above products, (so size \( O(s^4) \))
- All addition gates have fan-in at most \( s^2 \),
- All multiplication gates have fan-in at most 5, and
- If \( v_1 \) is a child of a \( \times \)-gate \( v \) in \( \Phi' \), then \( \deg(v_1) \leq \deg(v)/2 \).
Theorem ([Valiant-Skyum-Berkowitz-Rackoff])

If $\Phi$ is a size $s$ circuit computing an $n$-variate degree $d$ polynomial $f$, then there is a circuit $\Phi'$ computing $f$ with the following properties.

- Every gate of $\Phi'$ computes either $[u]$, $[u : v]$, or one of the above products, (so size $O(s^4)$)
- All addition gates have fan-in at most $s^2$,
- All multiplication gates have fan-in at most 5, and
- If $v_1$ is a child of a $\times$-gate $v$ in $\Phi'$, then $\deg(v_1) \leq \deg(v)/2$.

Hence, the depth of $\Phi'$ is $O(\log d)$. 

\[
[u] = \sum_{v \in \mathcal{F}_a} [u : v] \cdot [v_L] \cdot [v_R]
\]
\[
[u : w] = \sum_{v \in \mathcal{F}_a} \sum_{q \in \mathcal{F}_a} [u : v] \cdot [v : q] \cdot [q_L] \cdot [q_R] \cdot [v_R : w]
\]
First consequences of [VSBR]
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- A sized-$s$ circuit can be simulated by a formula of size $s^{O(\log d)}$. 

```
s
```

```
\rightarrow \binom{\text{poly}(s)}{d}

\rightarrow \text{formula: } \binom{\text{poly}(s)}{\log d}
```
First consequences of [VSBR]

- A sized-$s$ circuit can be simulated by a formula of size $s^{O(\log d)}$.
- Easy way to construct universal circuits.
Reducing to depth four

Can we reduce the depth further?
Reducing to depth four

Can we reduce the depth further?

**Theorem (Koiran)**

If \( f \) is computed by a circuit of size \( s \), then it is computed by a \( \sum \prod \sum \prod \) of size \( s^{O(\sqrt{d \log d})} \).
Reducing to depth four

Can we reduce the depth further?

**Theorem (Koiran)**

If $f$ is computed by a circuit of size $s$, then it is computed by a $\Sigma\Pi\Sigma\Pi$ of size $s^{O(\sqrt{d}\log d)}$.

**Lemma**

If $f$ is computed by an $ABP$ of size $s$, then it is computed by a $\Sigma\Pi\Sigma\Pi$ of size $s^{O(\sqrt{d})}$.
Reducing to depth four: starting from circuits
Reducing to depth four: starting from circuits

\[ \phi \]

Degree > \( \sqrt{d} \)

Degree \( \leq \sqrt{d} \)
Reducing to depth four: starting from circuits

Top

Degree $> \sqrt{d}$

Degree $\leq \sqrt{d}$
Reducing to depth four: starting from circuits

Degree > $\sqrt{d}$

Degree $\leq \sqrt{d}$

Size $\left( \frac{n+\sqrt{d}}{\sqrt{d}} \right)$ each
Reducing to depth four: starting from circuits

Degree $> \sqrt{d}$

Degree $\leq \sqrt{d}$

Size $(\frac{n + \sqrt{d}}{\sqrt{d}})$ each

**Lemma ([T.])**

*If the circuit has [VSBR] properties, then* $\deg(\text{Top}(z_1, \ldots, z_s)) \leq 15\sqrt{d}$
Reducing to depth four: starting from circuits

If the circuit has [VSBR] properties, then $\deg(\text{Top}(z_1, \ldots, z_s)) \leq 15\sqrt{d}$
Reducing to depth four: starting from circuits

**Theorem**

Equivalent depth-4 circuit of size

\[ s \left( \frac{n + \sqrt{d}}{n} \right) + \left( \frac{s + 15\sqrt{d}}{s} \right) = s^{O(\sqrt{d})} \]
Reducing to depth four: starting from circuits

Theorem

Equivalent depth-$4$ circuit of size

$$s \left( \binom{n + \sqrt{d}}{n} \right) + \left( \frac{s + 15\sqrt{d}}{s} \right) = s^{O(\sqrt{d})}$$
Reducing to depth four: starting from circuits

Theorem

Equivalent **homogeneous depth-4 circuit with bottom fan-in at most** $\sqrt{d}$

of size

$$s \left( \binom{n + \sqrt{d}}{n} \right) + \binom{s + 15\sqrt{d}}{s} = s^{O(\sqrt{d})}$$
Reducing to depth four: starting from circuits

Theorem

Equivalent homogeneous $\Sigma \Pi \Sigma \Pi^{[\sqrt{d}]}$ circuit of size

$$s\binom{n + \sqrt{d}}{n} + \binom{s + 15\sqrt{d}}{s} = sO(\sqrt{d})$$
Let’s start with \[ \text{[VSBR]} \]

\[
f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}
\]
Let’s start with [VSBR]

\[ f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5} \]

This is a \( \Sigma \Pi \Sigma \Pi^{[d/2]} \) circuit. We want to obtain a \( \Sigma \Pi \Sigma \Pi^{[t]} \) circuit.
Let’s start with [VSBR]

\[ f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5} \]

This is a \( \Sigma \Pi \Sigma \Pi^{[d/2]} \) circuit. We want to obtain a \( \Sigma \Pi \Sigma \Pi^{[t]} \) circuit. Each \( f_{ij} \) is also some \([u : v]\). Keep expanding terms of degree more than \( t \).
Let’s start with [VSBR]

\[
f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}
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Let’s start with [VSBR]

\[ f = \sum_{i=1}^{s} \left( \sum_{j=1}^{s} g_{j1} \cdots g_{j5} \right) \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5} \]

This is a $\Sigma \Pi \Sigma \Pi^{[d/2]}$ circuit. We want to obtain a $\Sigma \Pi \Sigma \Pi^{[t]}$ circuit. Each $f_{ij}$ is also some $[u : v]$. Keep expanding terms of degree more than $t$. 
Let's start with [VSBR]

\[ f = \sum_{i=1}^{s^2} f_{i1} \cdots f_{i9} \]

This is a $\Sigma\Pi\Sigma\Pi^{[d/2]}$ circuit. We want to obtain a $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit. Each $f_{ij}$ is also some $[u : v]$. Keep expanding terms of degree more than $t$. 
Let’s start with [VSBR]

\[ f = \sum_{i=1}^{s^3} f_{i1} \cdots f_{i13} \]

This is a \( \Sigma \Pi \Sigma \Pi^{[d/2]} \) circuit. We want to obtain a \( \Sigma \Pi \Sigma \Pi^{[t]} \) circuit. Each \( f_{ij} \) is also some \([u : v]\). Keep expanding terms of degree more than \( t \).
Let’s start with [VSBR]

\[ f = \sum_{i=1}^{s^4} f_{i1} \cdots f_{i17} \]

This is a $\Sigma \Pi \Sigma \Pi^{[d/2]}$ circuit. We want to obtain a $\Sigma \Pi \Sigma \Pi^{[t]}$ circuit. Each $f_{ij}$ is also some $[u : v]$. Keep expanding terms of degree more than $t$. 
[SV]'s proof

Let's start with [VSBR]

\[ f = \sum_{i=1}^{s^4} f_{i1} \cdots f_{i17} \]

This is a $\Sigma \Pi \Sigma \Pi^{[d/2]}$ circuit. We want to obtain a $\Sigma \Pi \Sigma \Pi^{[t]}$ circuit. Each $f_{ij}$ is also some $[u : v]$. Keep expanding terms of degree more than $t$.

How many iterations until all degrees are at most $t$?
Number of iterations

\[ g = \sum_{j=1}^{s} g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5} \]
Number of iterations

\[ g = \sum_{j=1}^{s} g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5} \]

**Observation**

*In each summand, at least two terms have degree at least \( t/8 \).*
Number of iterations

\[ g = \sum_{j=1}^{s} g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5} \geq t/5 \]

**Observation**

*In each summand, at least two terms have degree at least \( t/8 \).*
Number of iterations

\[ g = \sum_{j=1}^{s} g_j \cdot g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5} \]

Observation

*In each summand, at least two terms have degree at least \( t/8 \).*
Number of iterations

\[ g = \sum_{j=1}^{s} g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5} \]

\[ \geq \frac{t}{5} \cdot \frac{t}{8} \]

**Observation**

*In each summand, at least two terms have degree at least \( t/8 \).*

How many factors of degree at least \( t/8 \)?

\[ f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5} \]
Number of iterations

\[ g = \sum_{j=1}^{s} \left( g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5} \right) \geq \frac{t}{5} \cdot \frac{t}{8} \cdot \frac{t}{2} \cdot \frac{t}{4} \cdot \frac{t}{8} \]

Observation

*In each summand, at least two terms have degree at least \( t/8 \).*

How many factors of degree at least \( t/8 \)?

\[ f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5} \]
Number of iterations

\[ g = \sum_{j=1}^{s} g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5} \geq \frac{t}{5} \cdot \frac{t}{8} \cdot \frac{t}{2} \cdot \frac{t}{3} \cdot \frac{t}{4} \]

Observation

*In each summand, at least two terms have degree at least \( \frac{t}{8} \).*

How many factors of degree at least \( \frac{t}{8} \)?

\[ f = \sum_{i=1}^{s} \left( \sum_{j=1}^{s} g_{j1}g_{j2}g_{j3}g_{j4}g_{j5} \right) \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5} \]
Number of iterations

\[ g = \sum_{j=1}^{s} g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5} \]

\[ \geq t/5 \cdot \geq t/8 \]

**Observation**

*In each summand, at least two terms have degree at least \( t/8 \).*

How many factors of degree at least \( t/8 \)?

\[ f = \sum_{i=1}^{s} \left( \sum_{j=1}^{s} g_{j1} g_{j2} g_{j3} g_{j4} g_{j5} \right) \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5} \]
Number of iterations

\[ g = \sum_{j=1}^{s} g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5} \geq \frac{t}{5} \cdot \frac{t}{8} \]

Observation

In each summand, at least two terms have degree at least \( \frac{t}{8} \).

How many factors of degree at least \( \frac{t}{8} \)?

\[ f = \sum_{i=1}^{s^2} f_{i1} \cdot f_{i12} \cdot f_{i3} \cdot f_{i4} \cdots f_{i9} \]
Number of iterations

$$g = \sum_{j=1}^{s} g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5} \geq \frac{t}{5} \cdot \frac{t}{8}$$

Observation

*In each summand, at least two terms have degree at least $\frac{t}{8}$.*

How many factors of degree at least $\frac{t}{8}$? At most $8d/t$.

$$f = \sum_{i=1}^{s^2} f_{i1} \cdot f_{i12} \cdot f_{i3} \cdot f_{i4} \cdots f_{i9}$$
Number of iterations

\[ g = \sum_{j=1}^{s} g_{j1} \cdot \underbrace{g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5}}_{\geq t/5 \; \geq t/8} \]

Observation

*In each summand, at least two terms have degree at least \( t/8 \).*

How many factors of degree at least \( t/8 \)? At most \( 8d/t \).

\[ f = \sum_{i=1}^{s^2} f_{i1} \cdot f_{i12} \cdot f_{i3} \cdot f_{i4} \cdots f_{i9} \]

Final \( \Sigma \Pi \Sigma \Pi \left[ t \right] \) circuit has top fan-in at most \( s^{O(d/t)} \).
A better starting point?

Recall

If $f$ has a sized-$s$ circuit, then it has a $\Sigma \Pi \Sigma \Pi^{[\sqrt{d}]}$ of size $s^{O(\sqrt{d})}$.

$$f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}$$

If we start with a homogeneous formula, can we do better?
A better starting point?

Recall

If $f$ has a sized-$s$ circuit, then it has a $\Sigma \Pi \Sigma \Pi^{[\sqrt{d}]}$ of size $s^{O(\sqrt{d})}$.

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If we start with a homogeneous formula, can we do better? [Hrubes-Yehudayoff]: Yes!
A better starting point?

Recall

If \( f \) has a sized-\( s \) circuit, then it has a \( \Sigma \Pi \Sigma \Pi^{[\sqrt{d}]} \) of size \( s^{O(\sqrt{d})} \).

\[
f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}
\]

If we start with a homogeneous formula, can we do better?

[Hrubes-Yehudayoff]: Yes!

Lemma ([Hrubes-Yehudayoff])

\[
f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdots f_{i\ell} \quad \text{with} \quad \left( \frac{1}{3} \right)^{j} \cdot d < \deg(f_{ij}) \leq \left( \frac{2}{3} \right)^{j} \cdot d
\]
Lemma ([Hrubes-Yehudayoff])

\[ f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdots f_{i\ell} \quad \text{with} \quad \left( \frac{1}{3} \right)^j \cdot d < \text{deg}(f_{ij}) \leq \left( \frac{2}{3} \right)^j \cdot d \]
A better starting point?

Recall
If $f$ has a sized-$s$ circuit, then it has a $\Sigma \Pi \Sigma \Pi^{[\sqrt{d}]}$ of size $s^{O(\sqrt{d})}$.

Theorem (Saptharishi?)
If $f$ has a homogeneous sized-$s$ formula, then it has a homogeneous $\Sigma \Pi^{[\Omega(d \log t/t)]} \Sigma \Pi^{[\sqrt{d}]}$. 

Structural results
October 15th, 2021
24 / 35
A better starting point?

Recall

If $f$ has a sized-$s$ circuit, then it has a $\sum \prod \Sigma \Pi [\sqrt{d}]$ of size $s^{O(\sqrt{d})}$.

Theorem (Saptharishi?)

If $f$ has a homogeneous sized-$s$ formula, then it has a homogeneous $\sum \prod [\Omega(d \log t/t)] \sum \prod [\sqrt{t}]$.

Theorem (KOS)

If $f$ has a syntactically multilinear sized-$s$ circuit, then it has a $\sum \prod \Sigma \Sigma \Pi$ of size $2^{O(\sqrt{N \log s})}$.
Generalization to homogeneous depth-2Δ

Theorem

If \( f \) has a sized-\( \text{poly}(N) \) syntactically multilinear circuit, then it has a \( (\mathcal{\exists} \mathcal{\exists} \mathcal{\exists}) \) of size \( s \Omega(d^{1/3}) \).

Structural results

October 15th, 2021
Generalization to homogeneous depth-2Δ

**Theorem**

If \( f \) has a sized-\( s \) circuit, then it has a depth-2\( \Delta \) \( \sum \prod [O(d^{1/\Delta})] \sum \prod [O(d^{1/\Delta})] \cdots \sum \prod [O(d^{1/\Delta})] \) of size \( s^{O(\Delta \cdot d^{1/\Delta})} \).
**Generalization to homogeneous depth-2Δ**

**Theorem**

If $f$ has a sized-$s$ circuit, then it has a depth-$2Δ$ $\sum \prod [O(d^{1/Δ})] \sum \prod [O(d^{1/Δ})] \ldots \sum \prod [O(d^{1/Δ})]$ of size $s^{O(Δ \cdot d^{1/Δ})}$.

**Theorem**

If $f$ has a sized-$\text{poly}(N)$ syntactically multilinear circuit, then it has a $(\sum \prod)^Δ$ of size $s^{O(Δ \cdot (n/\log s)^{1/Δ})}$. 
Reduction to Depth-3 Circuits

(or, “can we do better if we allow the final circuit to be highly inhomogeneous?”)
Road map [GKKS]

\[ \sum \prod \prod \prod \]

circuits

\[ \sum \wedge \sum \wedge \sum \wedge \sum \]

circuits

\[ \sum \prod \prod \prod \]

circuits

App. of Ryser's formula [Saxena]'s duality trick

Only over \( Q \), \( R \) etc.

Heavily non-homogeneous

Structural results

October 15th, 2021
Road map [GKKS]

\[
\sum \prod \sum \prod \\
\text{circuits}
\]

App. of Ryser’s formula

\[
\sum \land \sum \land \sum \\
\text{circuits}
\]

\[
\sum \prod \sum \\
\text{circuits}
\]
Road map [GKKS]

\[ \sum \prod \sum \prod \prod \]
circuits

App. of Ryser’s formula

\[ \sum \land \sum \land \sum \land \sum \]
circuits

[Saxena]’s duality trick

\[ \sum \prod \sum \prod \prod \prod \]
circuits
Road map [GKKS]

\[ \sum \prod \sum \prod \text{circuits} \]

Only over \( \mathbb{Q}, \mathbb{R} \) etc.

App. of Ryser’s formula

\[ \sum \land \sum \land \sum \text{circuits} \]

Heavily non-homogeneous

[Saxena]'s duality trick

\[ \sum \prod \sum \text{circuits} \]
Step 1: $\Pi^d$ to $\Sigma^{2^d} \wedge^d \Sigma^d$
Step 1: $\prod^{[d]} \to \Sigma^{[2^d]} \land^{[d]} \Sigma^{[d]}$

Recall Ryser's formula:

$$\text{Perm}_d \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & \ddots & \vdots \\ x_{d1} & \cdots & x_{dd} \end{bmatrix} = \sum_{S \subseteq [d]} (-1)^{d-|S|} \prod_{i=1}^{d} \sum_{j \in S} x_{ij}$$
Step 1: $\Pi^{[d]}$ to $\sum^{[2^d]} \land [d] \sum^{[d]}$

Recall Ryser’s formula:

\[
\begin{bmatrix}
  x_1 & \ldots & x_d \\
  \vdots & \ddots & \vdots \\
  x_1 & \ldots & x_d \\
\end{bmatrix}
= \sum_{S \subseteq [d]} (-1)^{d-|S|} \prod_{i=1}^{d} \sum_{j \in S} x_j
\]
Step 1: $\prod^{[d]} \to \Sigma^{[2^d]} \wedge^{[d]} \Sigma^{[d]}$

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$$
\text{Perm}_d \begin{bmatrix}
    x_1 & \cdots & x_d \\
    \vdots & \ddots & \vdots \\
    x_1 & \cdots & x_d
\end{bmatrix} = \sum_{S \subseteq [d]} (-1)^{d - |S|} \left( \sum_{j \in S} x_j \right)^d
$$
Step 1: $\Pi^{[d]}$ to $\Sigma^{[2^d]} \wedge [d] \Sigma^{[d]}$

Recall Ryser’s formula:

$$d! \cdot x_1 \ldots x_d = \sum_{S \subseteq [d]} (-1)^{d-|S|} \left( \sum_{j \in S} x_j \right)^d$$
Step 1: $\Pi^{[d]}$ to $\Sigma^{[2^d]} \land [d] \Sigma^{[d]}$

[Fischer]:

$$d! \cdot x_1 \ldots x_d = \sum_{S \subseteq [d]} (-1)^{d - |S|} \left( \sum_{j \in S} x_j \right)^d$$
Step 1: $\Pi^d\text{ to } \Sigma^{2^d}\land [d] \Sigma^d$

[Fischer]:

$$d! \cdot x_1 \ldots x_d = \sum_{S \subseteq [d]} (-1)^{d-|S|} \left( \sum_{j \in S} x_j \right)^d$$

Structural results
October 15th, 2021
Step 1: $\Pi^{[d]} \rightarrow \Sigma^{[2^d]} \land^{[d]} \Sigma^{[d]}$
Step 1: $\Pi^d$ to $\Sigma^{2^d} \land [d] \Sigma^d$

\[
\Pi \rightarrow \Sigma \land \Sigma
\]

\[
\sum \prod \sum \prod \text{ of size } s \quad \rightarrow \quad \sum \land \sum \land \sum \text{ of size } 2^{O(\sqrt{d})} \cdot s
\]
Road map

\[ \sum \prod \sum \prod \]

\[ \text{circuits} \]

\[ \sum \bigwedge \sum \bigwedge \sum \bigwedge \sum \]

\[ \text{circuits} \]

\[ \sum \prod \sum \]

\[ \text{circuits} \]
Road map

\[
\sum \prod \sum \prod \sqrt{d} \text{circuits}
\]

\[
\sum \land \sum \land \sum \sqrt{d} \text{circuits}
\]

\[
\sum \prod \sum \text{circuits}
\]
Step 2: $\wedge \sum^{[a]} \wedge \sum^{[b]} \rightarrow \sum^{[\text{poly}(s,a,b)]} \Pi^{[sbd]} \sum^{[2]}$
Step 2: $\land^a\Sigma^s\land^b$ to $\Sigma^{\text{poly}(s,a,b)}\prod^{sbd}\Sigma^2$

$$T = (x_1^b + \cdots + x_s^b)^a$$
Step 2: \( \wedge^a [s] \vee [b] \) to \( \Sigma^{\text{poly}(s,a,b)} \prod [sbd] \Sigma^2 \)

\[
T = \left( x_1^b + \cdots + x_s^b \right)^a
\]

**Lemma ([Saxena])**

There exists univariate polynomials \( f_{ij} \)'s of degree at most \( a \) such that

\[
\ell^a = (y_1 + \cdots + y_s)^a = O(sa^2) \sum_{i=1}^s \prod_{j=1}^s f_{ij}(x_j)
\]
Step 2: $\wedge [a] \Sigma [s] \wedge [b]$ to $\Sigma [\text{poly}(s,a,b)] \Pi [sbd] \Sigma [2]$

$$T = \left( x_1^b + \cdots + x_s^b \right)^a$$

**Lemma ([Saxena])**

There exists univariate polynomials $f_{ij}$’s of degree at most $a$ such that

$$\ell^a = (y_1 + \cdots + y_s)^a = \sum_{i=1}^{O(sa^2)} \prod_{j=1}^s f_{ij}(x_j)$$

**Sketch of a proof by Gupta-Forbes-Shpilka**

$$P_y(t) = (1 + y_1 t) \cdots (1 + y_s t) = 1 + \ell t + (\text{higher degree terms}) \rightarrow s$$
Step 2: $\wedge [a] \sum [s] \wedge [b]$ to $\sum [\text{poly}(s,a,b)] \prod [sbd] \sum [2]$

$$T = \left(x_1^b + \cdots + x_s^b\right)^a$$

**Lemma ([Saxena])**

*There exists univariate polynomials $f_{ij}$’s of degree at most $a$ such that*

$$\ell^a = (y_1 + \cdots + y_s)^a = \sum_{i=1}^{O(sa^2)} \prod_{j=1}^s f_{ij}(x_j)$$

**Sketch of a proof by Gupta-Forbes-Shpilka**

$$P_y(t) - 1 = \ell t + \text{(higher degree terms)} \rightarrow s$$
Step 2: $\bigwedge^a[\sum^s\bigwedge^b]$ to $\sum^{\text{poly}(s,a,b)}\prod^{[sbd]}\sum^2$

$$T = \left(x_1^b + \cdots + x_s^b\right)^a$$

**Lemma ([Saxena])**

There exists univariate polynomials $f_{ij}$’s of degree at most $a$ such that

$$\ell^a = (y_1 + \cdots + y_s)^a = \sum_{i=1}^{O(sa^2)} \prod_{j=1}^s f_{ij}(x_j)$$

**Sketch of a proof by Gupta-Forbes-Shpilka**

$$(P_y(t) - 1)^a = \ell^a t^a + \text{(higher degree terms)} \rightarrow sa$$
Step 2: $\bigwedge[^a][\sum[^s]\wedge[^b]]$ to $\sum[^{\text{poly}(s,a,b)}]\prod[^{sbd}]\sum[^2]$

$$T = \left( x_1^b + \cdots + x_s^b \right)^a$$

**Lemma ([Saxena])**

There exists univariate polynomials $f_{ij}$'s of degree at most $a$ such that

$$\ell^a = (y_1 + \cdots + y_s)^a = O(sa^2) \sum_{i=1}^{s} \prod_{j=1}^{s} f_{ij}(x_j)$$

**Sketch of a proof by Gupta-Forbes-Shpilka**

$$(P_y(t) - 1)^a = \ell^a t^a + (\text{higher degree terms}) \rightarrow sa$$

Interpolate!

$(P_y(t) - 1)^a$ expanded is a sum of $(a + 1)$ product of univariates.
Step 2: $\wedge^a \sum^s \wedge^b$ to $\sum^{\text{poly}(s,a,b)} \prod^{\text{sbd}} \sum^2$

$$T = \left(x_1^b + \cdots + x_s^b\right)^a$$

$$(y_1 + \cdots + y_s)^a = \sum_{i}^{\text{poly}(s,a)} \prod_{j=1}^{s} f_{ij}(y_j)$$
Step 2: $\land^a [s] \land^b [s] \rightarrow \sum^{\text{poly}(s,a,b)} \prod^{\text{sbd}} \sum^{[2]}$

\[
T = \left(x_1^b + \cdots + x_s^b\right)^a
\]

\[
\left(x_1^b + \cdots + x_s^b\right)^a = \sum_{i}^{\text{poly}(s,a)} \prod_{j=1}^{s} f_{ij} \left(x_j^b\right)
\]
Step 2: $\wedge [a] \sum [s] \wedge [b]$ to $\sum [\text{poly}(s,a,b)] \prod [sbd] \sum [2]$

$$T = \left( x_1^b + \cdots + x_s^b \right)^a$$

$$\left( x_1^b + \cdots + x_s^b \right)^a = \sum_{i} \prod_{j=1}^{\text{poly}(s,a)} f_{ij} \left( x_j^b \right)$$

$$= \sum_{i} \prod_{j=1}^{\text{poly}(s,a)} \tilde{f}_{ij}(x_j)$$

where $\tilde{f}_{ij}(t) := f_{ij}(t^{\sqrt{d}})$
Step 2: $\land [a] \sum [s] \land [b]$ to $\sum [\text{poly}(s,a,b)] \prod [sbd] \sum [2]$

\[ T = \left( x_1^b + \cdots + x_s^b \right)^a \]

\[
\left( x_1^b + \cdots + x_s^b \right)^a = \sum_{i=1}^{\text{poly}(s,a)} \prod_{j=1}^{s} f_{ij} \left( x_j^b \right)
\]

\[
= \sum_{i=1}^{\text{poly}(s,a)} \prod_{j=1}^{s} \tilde{f}_{ij}(x_j)
\]

Note that $\tilde{f}_{ij}(t)$ is a univariate polynomial.
Step 2: $\bigwedge^{[a]} \Sigma [s] \bigwedge^{[b]}$ to $\Sigma [\text{poly}(s,a,b)] \prod [sbd] \Sigma [2]$

$$T = \left( x_1^b + \cdots + x_s^b \right)^a$$

$$\left( x_1^b + \cdots + x_s^b \right)^a = \sum_{i} \prod_{j=1}^{s} f_{ij} \left( x_j^b \right)$$

$$= \sum_{i} \prod_{j=1}^{s} \tilde{f}_{ij}(x_j)$$

Note that $\tilde{f}_{ij}(t)$ is a univariate polynomial that can be factorized over $\mathbb{C}$:

$$\tilde{f}_{ij}(t) = \prod_{k=1}^{ab} (t - \zeta_{ijk})$$
Step 2: $\land [a] \sum [s] \land [b] \text{ to } \sum [\text{poly}(s,a,b)] \prod [sbd] \sum [2]$

$$T = (x_1^b + \cdots + x_s^b)^a$$

$$\left( x_1^b + \cdots + x_s^b \right)^a = \sum_{i} \prod_{j=1}^{s} f_{ij} \left( x_j^b \right)$$

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$$\tilde{f}_{ij}(\ell_j) = \prod_{k=1}^{ab} (\ell_j - \zeta_{ijk})$$
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\[ T = \left( x_1^b + \cdots + x_s^b \right)^a \]

\[
\left( x_1^b + \cdots + x_s^b \right)^a = \sum_{i} \prod_{j=1}^{s} f_{ij} \left( x_j^b \right)
\]

\[
= \sum_{i} \prod_{j=1}^{s} \tilde{f}_{ij}(x_j)
\]

\[
= \sum_{i} \prod_{j=1}^{s} \prod_{k=1}^{ab} \left( x_j - \zeta_{ijk} \right)
\]
Step 2: $\wedge^a[s] \wedge^b$ to $\sum[\text{poly}(s,a,b)] \prod[sbd] \sum[2]$

$$T = \left(x_1^b + \cdots + x_s^b\right)^a$$

$$\left(x_1^b + \cdots + x_s^b\right)^a = \sum_{i=1}^{\text{poly}(s,a)} \prod_{j=1}^{s} f_{ij} \left(x_j^b\right)$$

$$= \sum_{i=1}^{\text{poly}(s,a)} \prod_{j=1}^{s} \tilde{f}_{ij}(x_j)$$

$$= \sum_{i=1}^{\text{poly}(s,a)} \prod_{j=1}^{s} \prod_{k=1}^{ab} \left(x_j - \zeta_{ijk}\right)$$

... a $\Sigma \Pi \Sigma$ circuit of $\text{poly}(s, a, b)$ size.
Step 2: $\land [a] \sum [s] \land [b]$ to $\sum [\text{poly}(s,a,b)] \prod [sbd] \sum [2]$ 

$$
T = \left( x_1^b + \cdots + x_s^b \right)^a
$$

$$
\left( x_1^b + \cdots + x_s^b \right)^a = \sum_{i=1}^{\text{poly}(s,a)} \prod_{j=1}^{s} f_{ij} \left( x_j^b \right)
$$

$$
= \sum_{i=1}^{\text{poly}(s,a)} \prod_{j=1}^{s} \tilde{f}_{ij}(x_j)
$$

$$
= \sum_{i=1}^{\text{poly}(s,a)} \prod_{j=1}^{s} \prod_{k=1}^{ab} \left( x_j - \zeta_{ijk} \right)
$$

... a $\Sigma \Pi \Sigma$ circuit of $\text{poly}(s, a, b)$ size and degree $sab$. 
Putting it together

general circuit of size $s$
Putting it together

A general circuit of size $s$ can be transformed into a sum of $\frac{\sqrt{d}}{2}$ circuits of size $\mathcal{O}(\sqrt{d})$. The transformation can be represented as:

$$\sum \prod \prod \prod \prod \prod$$

of size $s^{\mathcal{O}(\sqrt{d})}$
Putting it together

general circuit of size $s$

$\sum \prod \prod \prod \prod$ circuit of size $s^{O(\sqrt{d})}$

$\sum \land \sum \land \sum$ circuits of size $s^{O(\sqrt{d})} \cdot 2^{O(\sqrt{d})}$
Putting it together

general circuit of size $s$ \rightarrow \sum \prod \sum \prod \text{circuit of size } s^{O(\sqrt{d})}

\sum \prod \sum \prod \text{circuits of size } s^{O(\sqrt{d})}
Putting it together

![Diagram of putting it together](image-url)
Putting it together

Question: Where should one try to prove lower bounds?
Putting it together

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Putting it together

Question: Where should one try to prove lower bounds?
Putting it together

\[
\sum \prod \sum \prod \text{hom. circuit of size } s^{O(\sqrt{d})}
\]

Question: Where should one try to prove lower bounds?
Putting it together

- general hom. circuit of size $s$
- $\sum \prod \sum \prod^{\sqrt{d}}$ hom. circuit of size $s^{O(\sqrt{d})}$

$\sum \prod \sum$ non-hom. circuits of size $s^{O(\sqrt{d})}$

$\sum \bigwedge \sum \bigwedge \sum^{\sqrt{d}}$ hom. circuits of size $s^{O(\sqrt{d})}$

**Question:** Where should one try to prove lower bounds?
Other constants for the depth?

Recall
If $f$ has a sized-$s$ circuit, then it has a depth-$2\Delta$ \( \left( \sum \prod [O(d^{1/\Delta})] \right)^{\Delta} \) of size $s^{O(\Delta \cdot d^{1/\Delta})}$.
Other constants for the depth?

Recall
If \( f \) has a sized-\( s \) circuit,
then it has a depth-\( 2\Delta \left( \Sigma \prod [O(d^{1/\Delta})] \right)^\Delta \) of size \( s^{O(\Delta \cdot d^{1/\Delta})} \).

Theorem
If \( f \) has a sized-\( s \) circuit, then it has a depth-\( p \) circuit of size \( s^{O(p \cdot d^{1/(p-1)})} \).
Other constants for the depth?

Recall
If $f$ has a sized-$s$ circuit, then it has a depth-$2\Delta \left(\sum \prod [O(d^{1/\Delta})]\right)\Delta$ of size $s^{O(\Delta \cdot d^{1/\Delta})}$.

Theorem
If $f$ has a sized-$s$ circuit, then it has a depth-$p$ circuit of size $s^{O(p \cdot d^{1/(p-1)})}$.

Corollary
- $\text{Det}_n$ has a $\Sigma \Pi \Sigma \Pi$ of size $n^{O(3/\sqrt{n})}$.
- $\text{IMM}_{n,d}$ has a $\Sigma \Pi \Sigma \Pi$ of size $n^{O(3/\sqrt{d})}$.
- If $\text{Perm}_n$ needs $\Sigma \Pi \Sigma \Pi$ of size $n^{\omega(3/\sqrt{n})}$, then $\text{VP} \neq \text{VNP}$.
Back to the homogeneization (case of constant depth)

- All gates compute *homogeneous polynomials*.
- Hence, no gate can compute polynomials of degree larger than output.
- For circuits and ABPs, homogeneity can be assumed without loss of generality.
  - For formulas, probably not.
  - For constant depth formulas, certainly not.
Back to the homogeneization (case of constant depth)

- All gates compute *homogeneous polynomials*.
- Hence, no gate can compute polynomials of degree larger than output.
- For circuits and ABPs, homogeneity can be assumed without loss of generality.
  - For formulas, probably not.
  - For constant depth formulas, certainly not.

What happens if we allow some subexponential blow up?

**Theorem (Raz)**

If $f$ computed by a formula of size $s$, then it is computed by a homogeneous one of size $2^{O(d \log \log s)}$.
Back to the homogeneization (case of constant depth)

- All gates compute *homogeneous polynomials*.
- Hence, no gate can compute polynomials of degree larger than output.
- For circuits and ABPs, homogeneity can be assumed without loss of generality.
  For formulas, probably not.
  For constant depth formulas, certainly not.

**Theorem (Raz)**

If $f$ computed by a formula of size $s$, then it is computed by a *homogeneous* one of size $2^{O(d \log \log s)}$.
Back to the homogeneization (case of constant depth)

Theorem (GKKS)

If $f$ computed by a circuit of size $s$ and depth 3, then it is computed by a \textit{homogeneous} one of size $\text{poly}(s)2^{O(\sqrt{d})}$ and depth 5.
Back to the homogeneization (case of constant depth)

Theorem (GKKS)

If $f$ computed by a circuit of size $s$ and depth $3$, then it is computed by a homogeneous one of size $\text{poly}(s)2^{O(\sqrt{d})}$ and depth $5$.

Theorem (LST)

If $f$ computed by a circuit of size $s$ and depth $\Gamma$, then it is computed by a homogeneous one of size $\text{poly}(s)2^{O(\sqrt{d})}$ and depth $2\Gamma - 1$. 
Thank you.