Invariants for group actions, logic and complexity

Thomas SEILLER
(CNRS)

⚠ Not really about GCT

⚠ Not about algebraic geometry

1. Curry-Howard correspondence (proofs as programs).

logic	Computer Science
Formulas $\mathrm{Nat} := \forall X.\ \underbrace{(X \to X)}_{f} \to \underbrace{(X \to X)}_{f^k}$	Types integers (unary rep.)
Proofs (cut-free) $\vdash \mathrm{Nat}$	Programs $\bar{k} : f \mapsto f^k$
Normalisation $\dfrac{\overset{\bar{n}}{\vdash \mathrm{Nat}} \qquad \overset{P}{\mathrm{Nat} \vdash \mathrm{Nat}}}{\vdash \mathrm{Nat}}\ cut$ $\rightsquigarrow$ length $\overset{\bar{m}}{\vdash \mathrm{Nat}}$ without cuts SLL	Program execution $P(\bar{n})$ length $\rightsquigarrow$ $\bar{m}$ $=$ $\Rightarrow$ bounded by poly.

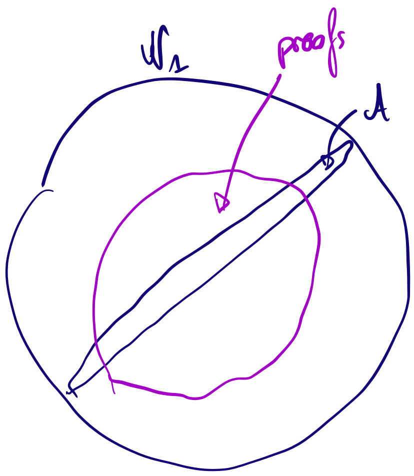
Implicit computational complexity.

2. von Neumann algebras.

$\mathcal{N} \subseteq \mathcal{B}(\mathbb{H})$ ← Hilbert space

sub-algebra closed for WOT.

bounded linear op.



$$\mathcal{M}_2(\mathbb{C}) \longrightarrow \mathcal{M}_4(\mathbb{C}) \longrightarrow \cdots$$
$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \longmapsto \cdots$$

fix $\mathcal{A} \subseteq \mathcal{N}$ $\Rightarrow$ embed proofs.

proof normalisation $\Rightarrow$ functional equation.

Theorem ('06 ?) Girard — the solution to the equation extends to $\mathcal{N}_1$

$\mathcal{A} \subseteq \mathcal{N}$ + equation = model of computation

Changing $\mathcal{A} \subseteq \mathcal{N}$ $\Rightarrow$ change in the expressivity of the logic

$\Rightarrow$ change in the complexity class it captures.

Question:

$\mathcal{A}_1 \subseteq \mathcal{N}_1$
$\mathcal{A}_2 \subseteq \mathcal{N}_2$ if they characterize the same complexity class, are they equivalent?

3. Group action

Group measure space construction (Murray, von Neumann)

$$G \curvearrowright (X,\mu) \qquad \text{measure preserving maps.}$$

(X,μ) ← measure space

$$L^\infty(X,\mu) = \{ f : X \to \mathbb{C} \mid \|f\|_\infty < \infty \}$$

$$L^\infty(X,\mu) \text{ abelian vNa } \subseteq B(L^2(X,\mu)).$$

$$G \curvearrowright L^\infty(X,\mu) \rightsquigarrow \text{crossed product } L^\infty(X,\mu) \rtimes G = M$$

$$L^\infty(X,\mu) \subseteq L^\infty(X,\mu) \rtimes G, \qquad A \subseteq M$$

Monoid actions $M \curvearrowright X$ as a model of computation.

$$X = \{0,1,\ast\}^{\mathbb{Z}}_{\ast}$$

right : $X \to X$
$(a_i)_i \mapsto (b_i)_i \qquad b_i = a_{i+1}$

left.

write$_0$: $X \to X$
$(a_i)_i \mapsto (b_i)_i \qquad b_0 = 0$, $b_i = a_i$ otherwise

Program : "graphing" (generalises dynamical systems).

deterministic $\quad X \times D \xrightarrow{f} X \times D$

D ← states

f s.t. $G(f) = \{(x, f(x))\} \subseteq \{(x, m(x)) \mid m \in M, x \in X\}$

Principle: $\alpha : M \curvearrowright X$

↳ logical system

↳ complexity class $\quad f : x \to x \;:\; \text{Nat} \to \text{Bool}$

$CC(\alpha)$

Questions

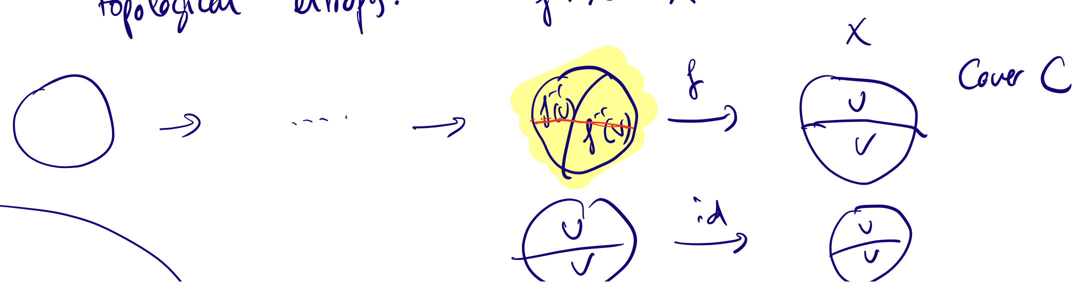
$CC[\alpha] = CC[\beta] \overset{?}{\Longrightarrow} \alpha \sim_{O.E.} \beta$

? orbit equivalence

$CC[\alpha]$	Deterministic	ND
α_1	Reg.	Reg
α_2	D_2	N_2
$\vdots$	$\vdots$	$\vdots$
α_i	D_i	N_i
$\vdots$	$\vdots$	$\vdots$
α_ω	LogSpace	NL
β	Ptime	Ptime
γ	Ptime	NP

4. Lower bounds & entropy.

topological entropy. $\quad f : X \to X$



lim

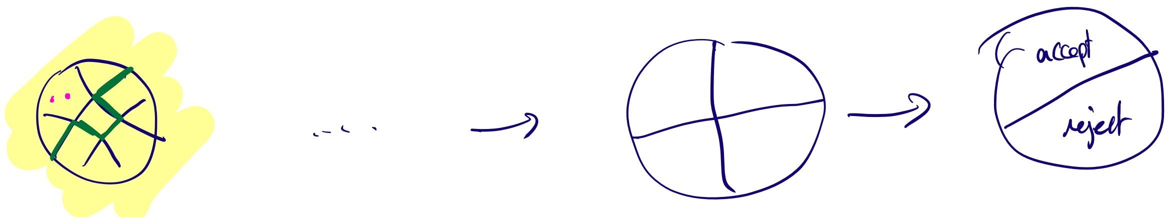
$\frac{1}{n} \log \left(\#\text{ smallest cover} \right)$

$= h(f, C)$

smallest cover

entropy $= \sup_{\text{finite covers } C} h(f, C)$

M machine $\rightsquigarrow$ $[M]$ dynamical system.



- decomposition of X in a number of "cells" $\leq C(h([M]))$
- bound on the degree of the polynomial equations
  - $\hookrightarrow$ bounds on $\beta_0(\text{cell})$

Steele & Yao '82 $\rightsquigarrow$ Ben'Or '83

Cucker '92

Mulmuley '99 $\rightsquigarrow$ extends the result with $\%$, $\sqrt{\ }$ (on $\mathbb{R}$).

5.

$$CC[\alpha] = CC[\beta] \overset{?}{\Longrightarrow} \alpha \sim_{o.E.} \beta$$

(?)

$$\alpha \sim_{conj} \beta$$

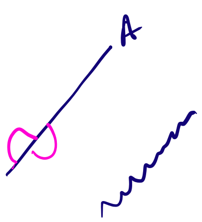
topological entropy
is an invariant

Orbit equivalence (groups) : invariants known (ℓ^2-Betti numbers Gaboriau)

$A \subseteq M_n(\mathbb{C})$ ⇝ $\mathcal{N}(A)'' = M_n(\mathbb{C})$

max. ab.
sub.-alg.

$$A \subseteq \mathcal{N} \rightsquigarrow \begin{cases} \mathcal{N}(A)'' = \mathcal{N} & \text{(Cartan subalg.)} \\ \mathcal{N}(A)'' = R \subseteq \mathcal{N} & \\ \mathcal{N}(A)'' = A & \text{(singular)} \end{cases}$$



$\pi \sim \pi'$ iff $\exists u$ unitary $\in \mathcal{N}$
$u^* \pi u = \pi'$

$\mathcal{N} \subseteq B(H)$

$\cancel{\exists}$ u

π, π' proj. on ∞-dim space
$\exists u : \pi \to \pi' \in B(H)$

tr/dim

$A \to \begin{pmatrix} A & \cdot \\ \cdot & A \end{pmatrix}$

finite $\not\exists$ $\pi' \lneq \pi$ $\pi' \sim \pi$ ⇝ trace.
⇝ Fuglede-Kadison det.

$$\det(\exp(A)) = \exp(\operatorname{tr}(A))$$