Implementing geometric complexity theory: On the separation of orbit closures via symmetries

Christian Ikenmeyer
joint work with Umangathan Kandasamy
1. Multiplicity obstructions

2. Fundamental invariant: the connection between orbit and its closure

3. Proof idea: Highest weight vectors via Young tableaux
Multiplicity obstructions

Fundamental invariant: the connection between orbit and its closure

Proof idea: Highest weight vectors via Young tableaux
Obstructions (Mulmuley and Sohoni (2001,2008))

In GCT we try to prove “variety noninclusions”: \( Y_m \not\subseteq Z_m \)

(1) Classically: \( \text{GL}_{p(m)}^2 \cdot (x_0^{p(m)} - m \text{ per}_m) \not\subseteq \text{GL}_{p(m)}^2 \cdot \det_{p(m)} \)

(2) Without padding: \( \text{GL}_{mp(m)}^2 \cdot \text{per}_m \not\subseteq \text{GL}_{mp(m)}^2 \cdot \text{imp}_{mp(m)}^m \)

(3) This talk: \( \text{GL}_m \cdot (x_1^m + x_2^m + \cdots + x_m^m) \not\subseteq \text{GL}_m \cdot (x_1 x_2 \cdots x_m) \) ← much simpler, polystable

Let \( G := \text{GL}_m \).

If \( \overline{G}p \subseteq \overline{G}q \), then

\[
\begin{array}{c}
\mathbb{C}[\overline{G}q]_\delta \\
\oplus \lambda \{ \lambda \}^{\oplus z_\lambda} \\
\end{array} \xrightarrow{\text{\|}} \begin{array}{c}
\mathbb{C}[\overline{G}p]_\delta \\
\oplus \lambda \{ \lambda \}^{\oplus y_\lambda}
\end{array}
\]

Schur’s lemma: \( \forall \lambda \) we have \( z_\lambda \geq y_\lambda \).

If \( z_\lambda < y_\lambda \), then \( \lambda \) is a multiplicity obstruction. If also \( z_\lambda = 0 \), then \( \lambda \) is an occurrence obstruction.

Theorem (with Bürgisser and Panova, 2016)

Occurrence obstructions cannot prove superpolynomial lower bounds for (1).

Hope (Mulmuley and Sohoni): For orbit closures of points that are characterized by their symmetries: obstructions should be obtainable from the symmetries.
Hope for multiplicities

**Definition (characterized by stabilizer)**

A point $p$ is **characterized by its stabilizer** $H$, if for every point $q$ with $H \leq \text{stab}_G(q)$ we have $q \in C_p$.

Many points are characterized by their stabilizer: $x_1 \cdots x_m$, $x_1^m + \cdots + x_m^m$, $\det D$, $\text{per}_D$, $\text{imp}^{(m)}_D$, ...

**Proposition**

If a point that is characterized by its stabilizer, each of the following pieces of information is sufficient to determine the others:

1. The orbit $Gp$
2. The orbit closure $\overline{Gp}$
3. The stabilizer of $p$ up to conjugation

**Proposition**

For a polystable point, under technical assumptions ($G$ must be a compact Lie group and the stabilizer $H$ must be connected and $\mathbb{C}^m$ must be an irreducible $H$-representation) we can enlarge this list [Yu 2016], based on [Larsen, Pink 1990]:

4. The multiplicities in the coordinate ring of $Gp$
5. The multiplicities in the coordinate ring of $\overline{Gp}$
GL_m \cdot p is an algebraic variety, GL_m \cdot p \subseteq \overline{GL_m \cdot p} open in its closure.

\[ C[GL_m \cdot p] : \text{ring of regular functions on the orbit } GL_m \cdot p. \text{ For example } a^4/(b^2 - 4ac) \in C[GL_2(x_1)^2]. \]

\[ C[GL_m \cdot p] \subseteq C[GL_m \cdot p] \text{ subring. Therefore } \text{mult}_\lambda C[GL_m \cdot p] \leq \text{mult}_\lambda C[GL_m \cdot p]. \]

Let \( H \leq GL_m \) be the stabilizer of \( p \).

- For \( p = x_1^D + \cdots + x_m^D \) we have \( H = C_D \wr S_m \).
- For \( p = x_1 \cdots x_m \) we have \( H = ST_m \rtimes S_m \).
- For \( p = \det D, m = D^2 \), we have \( H = ((GL_D \times GL_D)/\mathbb{C}^\times) \rtimes S_2 \) (Frobenius 1896).

### Algebraic Peter-Weyl theorem

\[ GL_m \cdot p = GL_m / H \text{ via } gp \mapsto gH. \text{ Have a } GL_m \text{-action on } GL_m / H \text{ via } g'(gH) := (g'g)H. \]

\[ C[GL_m \cdot p] = C[GL_m]^H \cong \bigoplus_\lambda \{ \lambda \} \otimes \{ \lambda \}^H. \text{ Hence } \text{mult}_\lambda C[GL_m \cdot p] = \dim(\{ \lambda \})^H. \]

\( \dim(\{ \lambda \})^H \) can be calculated using the stabilizer and representation theoretic branching rules:

- For \( p = \det D \) we have \( \dim(\{ \lambda \})^H = sk(\lambda, D \times \delta) \), a symmetric rectangular Kronecker coefficient.
- For \( p = x_1 \cdots x_m \) we have \( \dim(\{ \lambda \})^H = a_\lambda(m, \delta) \), a plethysm coefficient.

### Theorem (with Panova), proof via classical representation theoretic branching coefficients

For \( p = x_1^D + \cdots + x_m^D \) we have \( \dim(\{ \lambda \})^H = \sum_{\varrho = m \delta} b(\lambda, \varrho, D, \delta) \) with

\[ b(\lambda, \varrho, D, \delta) := \sum_{\mu_1, \mu_2, \ldots, \mu_\delta} c^\lambda_{\mu_1, \mu_2, \ldots, \mu_\delta} \prod_{i=1}^\delta a_{\mu_i}(\hat{\varrho}_i, iD). \]

and \( \hat{\varrho}_i = \# \{ j \mid i = \varrho_j \} \) (i.e., the frequency notation of \( \varrho \)).
<table>
<thead>
<tr>
<th>$x_0^{p(m)-m} \text{per}<em>m \text{ vs det}</em>{p(m)}$</th>
<th>occurrence obstructions</th>
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<th>power sum vs prod of lin forms in few variables</th>
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- All obstructions so far require an evaluation of a function at the “hard problem” (permanent, MaMu, etc).
- This is not intended in the Mulmuley-Sohoni papers.
- [I-Kandasamy 2020] does **not** require an evaluation. We show a tight connection

$$\mathbb{C}[\text{GL}_m(x_1^m + \cdots + x_m^m)] \leftrightarrow \mathbb{C}[\text{GL}_m(x_1^m + \cdots + x_m^m)].$$
Theorem [I, Kandasamy; STOC 2020]

Let $m \geq 3$. Let $Z_m := \text{GL}_m(x_1 x_2 \cdots x_m)$. Let $Y_m := \text{GL}_m(x_1^m + x_2^m + \cdots + x_m^m)$. Let $\lambda := (4m, 2m, 2m, 2m, \ldots, 2m)$. Then

$$a_\lambda(m + 1, m) \geq 3 > \text{mult}_\lambda(\mathbb{C}[Y_m]) = 2 > 1 \geq \text{mult}_\lambda(\mathbb{C}[Z_m])^{m = \text{prime} \pm 1} > 0.$$

Therefore

- $Y_m \not\subseteq Z_m$.

and hence $x_1^m + \cdots + x_m^m$ is not a product of homogeneous linear polynomials.

Caveat: Just the non-inclusion on its own is only barely nontrivial as a result.

The bounds are derived from the symmetry groups of $x_1^m + \cdots + x_m^m$ and $x_1 \cdots x_m$. 
Theorem [I, Kandasamy; STOC 2020]

Let \( m \geq 3 \). Let \( Z_m := \text{GL}_m(x_1x_2 \cdots x_m) \). Let \( Y_m := \text{GL}_m(x_1^m + x_2^m + \cdots + x_m^m) \). Let \( \lambda := (4m, 2m, 2m, 2m, \ldots, 2m) \). Then

\[
\lambda := (4m, 2m, 2m, 2m, \ldots, 2m) \text{ with } m-1 \text{ many}
\]

\[
a_\lambda(m + 1, m) \geq 3 > \text{mult}_\lambda(\mathbb{C}[Y_m]) = 2 > 1 \geq \text{mult}_\lambda(\mathbb{C}[Z_m]) \text{ for } m = \text{prime} \pm 1 > 0.
\]

Therefore

\[
Y_m \not\subseteq Z_m.
\]

and hence \( x_1^m + \cdots + x_m^m \) is not a product of homogeneous linear polynomials.

We have \( \text{mult}_\lambda(\mathbb{C}[\text{GL}_m(x_1 \cdots x_m)]_\delta) \leq \text{mult}_\lambda(\mathbb{C}[\text{GL}_m(x_1 \cdots x_m)]_\delta) = a_\lambda(m, \delta) \leq K(\lambda, m \times \delta) = \text{number of semistandard tableaux of shape } \lambda \text{ with numbers } 1, \ldots, m, \text{ each exactly } \delta \text{ times.}
\]

For the \( \lambda \) in the theorem there is only 1 such tableau!

Example \( (m = 4) \):

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]
1 Multiplicity obstructions

2 Fundamental invariant: the connection between orbit and its closure

3 Proof idea: Highest weight vectors via Young tableaux
Polystability: The fundamental invariant connects $\mathbb{C}[G_p]$ and $\mathbb{C}[G_p]$

Let $G := \text{GL}_m$.

- These points are $\text{SL}_m$-polystable, i.e., their $\text{SL}_m$-orbit is closed (Hilbert-Mumford-Luna-Kempf):
  - $x_1 \cdots x_m$
  - $x_1^m + \cdots + x_m^m$
  - $\det_D$ for $m = D^2$
  - $\text{per}_D$ for $m = D^2$

Theorem (B"urgisser-I)

If $p$ is polystable, then

- there is an $\text{SL}_m$-invariant $\Phi \in \mathbb{C}[G_p]$ of smallest degree.
- $\Phi$ cuts out the boundary $G_p \setminus G_p \subseteq G_p$.
- The ring $\mathbb{C}[G_p]$ is the localization $\mathbb{C}[G_p]_{\Phi}$.

$\Phi$ is called the **fundamental invariant**.

- We know $\Phi$ for $x_1^D + \cdots + x_m^D$ for even $D$. For odd $D$ we need \((\frac{2D}{D}) \geq 2m\).
- For $x_1 \cdots x_m$ we know it for all cases in which the Alon-Tarsi conjecture (Latin squares) is true.
Theorem [I, Kandasamy; STOC 2020]

Let $m \geq 3$. Let $Z_m := \text{GL}_m(x_1 x_2 \cdots x_m)$. Let $Y_m := \text{GL}_m(x_1^m + x_2^m + \cdots + x_m^m)$. Let

\[
\lambda := (4m, 2m, 2m, 2m, \ldots, 2m). \text{ Then}
\]

\[
a_{\lambda}(m + 1, m) \geq 3 > \text{mult}_{\lambda}(\mathbb{C}[Y_m]) = 2 > 1 \geq \text{mult}_{\lambda}(\mathbb{C}[Z_m]) > 0.
\]

Let $p = x_1^m + \cdots + x_m^m$ so $G_p = Y_m$.

\[\text{mult}_{\lambda}\mathbb{C}[G_p] = \sum_{\rho \vdash m} b(\lambda, \rho, D, \delta) = 2, \text{ but that is only an upper bound for mult}_{\lambda}\mathbb{C}[G_p].\]

Main Technical Theorem

- If $D$ is even, let $e_{\rho} := \sum_{i=1}^{m} \left\lfloor \frac{\rho_i}{D-2} \right\rfloor$. If $D$ is odd, let $e_{\rho} := \sum_{i=1}^{m} 2\left\lfloor \frac{\rho_i}{2(D-2)} \right\rfloor$.
- Let $\lambda \vdash_m \delta D$.
- Let $k$ be the number of $m \times m$ blocks in $\lambda$.
- Let $d := \delta - mk$.
- Let $e := \max\{e_{\rho} \mid \rho \vdash_m d\}$.

\[
\text{If } k \geq e, \text{ then } \text{mult}_{\lambda}\mathbb{C}[G_p] = \text{mult}_{\lambda}\mathbb{C}[G_p].
\]

- $\lambda = (4m, 2m, 2m, \ldots, 2m)$
- $k = 2, \quad D = m, \quad \delta = 2m^2 + 2m$
- $d = \delta - mk = 2m^2$
- $e = \max\{e_{(2)}, e_{(1,1)}\} = \max\{1, 2\} = 2$. 

Christian Ikenmeyer

12
Multiplicity obstructions

Fundamental invariant: the connection between orbit and its closure

Proof idea: Highest weight vectors via Young tableaux
If $D$ is even, let $e_\varrho := \sum_{i=1}^m \lceil \frac{\varrho_i}{D-2} \rceil$. If $D$ is odd, let $e_\varrho := \sum_{i=1}^m 2 \lceil \frac{\varrho_i}{2(D-2)} \rceil$. Let $\lambda \vdash_m \delta D$. Let $k$ be the number of $m \times m$ blocks in $\lambda$. Let $d := \delta - mk$. Let $e := \max \{ e_\varrho \mid \varrho \vdash_m d \}$. If $k \geq e$, then $\text{mult}_\lambda \mathbb{C}[Gp] = \text{mult}_\lambda \mathbb{C}[Gp]$.

A vector $f$ in a $\text{GL}_m$-representation $\mathcal{V}$ is called a **highest weight vector** (HWV) of weight $\lambda$, if

- $\text{diag}(\alpha_1, \ldots, \alpha_m) f = \alpha_1^{\lambda_1} \cdots \alpha_m^{\lambda_m} f$

- $\begin{pmatrix} 1 & * \\ 0 & \cdots & 1 \end{pmatrix} f = f$

The highest weight vectors of weight $\lambda$ form a linear subspace $\text{HWV}_\lambda(\mathcal{V}) \subseteq \mathcal{V}$.

- $\text{mult}_\lambda(\mathcal{V}) = \dim \text{HWV}_\lambda(\mathcal{V})$
HWVs can be understood in terms of tableaux

A **Young tableau** or just **tableau** $T : \lambda \to \mathbb{N}$ is an assignment of numbers to the boxes of $\lambda$:

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The **content** of a tableau $T$ is the list: (number of 1s in $T$, number of 2s in $T$, …).

A Young tableau is called **semistandard** if each column is increasing from top to bottom and every row is nondecreasing from left to right.

The **superstandard tableau** $S_\lambda$ of shape $\lambda$ has only entries $i$ in row $i$:

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**Quotient out by the vector space of shuffle relations (Grassmann-Plücker relations)**

In the vector space of formal linear combinations of Young tableaux there is the linear subspace $K(\lambda)$, which is spanned by

- $T + T'$, where $T'$ is a tableau that arises from $T$ by switching two numbers within one column.
- $T - \Sigma S$, where for two fixed columns $j, j'$ and a fixed number of entries $k$ the sum is over all tableaux $S$ that arise from $T$ by exchanging the top $k$ entries in column $j$ with any $k$ entries in column $j'$, preserving the internal vertical order.

The action on the space of linear combinations of tableaux is defined via multilinear expansion.

**Example:**

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 + 2 + 1 + 2 \\ 2 + 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$
**Averaging**

- Let $T : \lambda \rightarrow \{1, \ldots, \delta\}$ be a tableau.
- Let $\varphi : \{1, \ldots, \delta\} \rightarrow \{1, \ldots, m\}$ be a map.
- Define $\varphi T := \varphi(T)$ naively.

Example:

\[
\begin{bmatrix}
\varphi(1) = 2 \\
\varphi(2) = 2 \\
\varphi(3) = 1
\end{bmatrix}
\text{ implies } \varphi \left(\begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 \end{bmatrix}.
\]

The "orbit average":

\[
M_{\delta,m} T := \sum_{\varphi : \{1, \ldots, \delta\} \rightarrow \{1, \ldots, m\}} \varphi T
\]

Example:

\[
M_{3,2} \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 1 & 2 \\ 1 & 1 \\ 2 \end{bmatrix} = 0
\]

The symmetrization:

\[
P_{m} T := \sum_{\pi \in \mathcal{S}_{m}} \pi T
\]

Example:

\[
P_{3} \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 & 1 & 3 \\ 2 & 2 & 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 & 2 & 1 \\ 3 & 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 3 & 2 \\ 2 & 2 & 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 3 & 1 \\ 3 & 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 & 1 & 2 \\ 3 & 3 & 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 \\ 1 \end{bmatrix} = 0
\]
Main Technical Theorem

- If $D$ is even, let $e_\varrho := \sum_{i=1}^{\varrho} \left\lceil \frac{\varrho}{D-2} \right\rceil$. If $D$ is odd, let $e_\varrho := \sum_{i=1}^{\varrho} 2\left\lceil \frac{\varrho}{2(D-2)} \right\rceil$.
- Let $\lambda \vdash m \delta D$.
  - Let $k$ be the number of $m \times m$ blocks in $\lambda$.
  - Let $d := \delta - mk$.
- Let $e := \max\{e_\varrho \mid \varrho \vdash m \delta d\}$.

If $k \geq e$, then $\operatorname{mult}_\lambda \mathbb{C}[Gp] = \operatorname{mult}_\lambda \mathbb{C}[Gp]$.

Theorem (HWVs in the coordinate ring of the orbit closure)

The vector space $\operatorname{HWV}_\lambda(\mathbb{C}[Gp]_\delta)$ is generated by the functions

$$ g \mapsto \langle S_\lambda, gM_{\delta,m}T \rangle,$$

where $T$ runs over all semistandard tableaux of shape $\lambda$ in which each entry $1, \ldots, \delta$ appears exactly $D$ times.

Theorem (HWVs in the coordinate ring of the orbit)

The vector space $\operatorname{HWV}_\lambda(\mathbb{C}[Gp]_\delta)$ decomposes into a direct sum of vector spaces $\operatorname{HWV}_\lambda(\mathbb{C}[Gp]_\delta) = \bigoplus_{\varrho \vdash m \delta} \mathcal{W}_\varrho$, and each $\mathcal{W}_\varrho$ is generated by the functions

$$ g \mapsto \langle S_\lambda, gP_mS \rangle,$$

where $S$ runs over all semistandard tableaux $S$ of shape $\lambda$ and content $\varrho D$. 
Theorem (HWVs in the coordinate ring of the orbit)

The vector space \( \text{HWV}_\lambda(\mathbb{C}[Gp]_\delta) \) decomposes into a direct sum of vector spaces \( \text{HWV}_\lambda(\mathbb{C}[Gp]_\delta) = \bigoplus_{\varphi} \text{HWV}_\varphi \), and each \( \varphi \) is generated by the functions
\[
g \mapsto \langle S_\lambda, gP_mS \rangle,
\]
where \( S \) runs over all semistandard tableaux of shape \( \lambda \) and content \( \varphi D \).

Example: \( D = 8 \), \( \lambda = (24, 8, 8, 8, 8, 8, 8, 8) \),

\[
S = \begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\end{array}
\]

\( \varphi = (3, 1, 1, 1, 1, 1, 1) \)

\[
P_8 S = 7! \left( \begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\end{array} + \cdots + \begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\end{array} \right)
\]
we construct

we construct

Proof of the technical theorem: This works for all \( \lambda \) with enough full \( m \times m \) blocks.
Main Technical Theorem

- If $D$ is even, let $e_\varrho := \sum_{i=1}^{m} \left\lceil \frac{\varrho_i}{D-2} \right\rceil$. If $D$ is odd, let $e_\varrho := \sum_{i=1}^{m} 2\left\lceil \frac{\varrho_i}{2(D-2)} \right\rceil$.
- Let $\lambda \vdash_m \delta D$.
  - Let $k$ be the number of $m \times m$ blocks in $\lambda$.
  - Let $d := \delta - mk$.
- Let $e := \max\{e_\varrho \mid \varrho \vdash_m d\}$.

If $k \geq e$, then $\text{mult}_\lambda \mathbb{C}[Gp] = \text{mult}_\lambda \mathbb{C}[Gp]$.

Theorem (HWVs in the coordinate ring of the orbit closure)

The vector space $\text{HWV}_{\lambda}(\mathbb{C}[Gp], \delta)$ is generated by the functions

$$g \mapsto \langle S_\lambda, gM_\delta, mT \rangle,$$

where $T$ runs over all semistandard tableaux of shape $\lambda$ in which each entry $1, \ldots, \delta$ appears exactly $D$ times.

Theorem (HWVs in the coordinate ring of the orbit)

The vector space $\text{HWV}_{\lambda}(\mathbb{C}[Gp], \delta)$ decomposes into a direct sum of vector spaces $\text{HWV}_{\lambda}(\mathbb{C}[Gp], \delta) = \bigoplus_{\varrho \vdash_m \delta} \mathcal{W}_\varrho$, and each $\mathcal{W}_\varrho$ is generated by the functions

$$g \mapsto \langle S_\lambda, gP_mS \rangle,$$

where $S$ runs over all semistandard tableaux $S$ of shape $\lambda$ and content $\varrho D$. 

Christian Ikenmeyer
Theorem [I, Kandasamy; STOC 2020]

Let $m \geq 3$. Let $Z_m := \text{GL}_m(x_1 x_2 \cdots x_m)$. Let $Y_m := \text{GL}_m(x_1^m + x_2^m + \cdots + x_m^m)$. Let

$$\lambda := (4m, 2m, 2m, 2m, \ldots, 2m).$$

Then

$$a_\lambda(m + 1, m) \geq 3 > \text{mult}_\lambda(\mathbb{C}[Y_m]) = 2 > 1 \geq \text{mult}_\lambda(\mathbb{C}[Z_m])^{m=\text{prime}\pm 1} > 0.$$
Summary

- The power sum does not factor as a product of linear forms.
- This is shown via explicit multiplicity obstructions $\lambda = (4m, 2m, 2m, \ldots, 2m)$.
- Not occurrence obstructions (in infinitely many cases).
- Not vanishing ideal occurrence obstructions.
- Both multiplicities are obtained via representation theoretic branching formulas.
- This is achieved by proving a close connection between $\mathbb{C}[Gp]$ and $\mathbb{C}[\overline{Gp}]$, $p = x_1^m + \cdots + x_m^m$.

Thank you for your attention!