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Implementing geometric complexity theory: On the separation of orbit closures via symmetries

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Obstructions (Mulmuley and Sohoni (2001,2008))

In GCT we try to prove "variety noninclusions": $Y_m \not\subseteq Z_m$

(1) Classically: $\overline{\mathsf{GL}_{p(m)^2} \cdot (x_0^{p(m)-m} \mathrm{per}_m)} \not\subseteq \overline{\mathsf{GL}_{p(m)^2} \cdot \det_{p(m)}}$

(2) Without padding: $\overline{\mathsf{GL}_{mp(m)^2} \cdot \mathrm{per}_m} \not\subseteq \overline{\mathsf{GL}_{mp(m)^2} \cdot \mathrm{imp}^{(m)}_{mp(m)^2}}$

(3) This talk: $\overline{\mathsf{GL}_m \cdot (x_1^m + x_2^m + \cdots + x_m^m)} \not\subseteq \overline{\mathsf{GL}_m \cdot (x_1 x_2 \cdots x_m)}$ $\leftarrow$ much simpler, polystable

Let $G := \mathsf{GL}_m$.

If $\overline{Gp} \subseteq \overline{Gq}$, then

$$\begin{array}{ccc} \mathbb{C}[\overline{Gq}]_\delta & \longrightarrow\!\!\!\!\rightarrow & \mathbb{C}[\overline{Gp}]_\delta \\ \| & & \| \\ \bigoplus_\lambda \{\lambda\}^{\oplus z_\lambda} & & \bigoplus_\lambda \{\lambda\}^{\oplus y_\lambda} \end{array}$$

Schur's lemma: $\forall \lambda$ we have $z_\lambda \geq y_\lambda$.

If $z_\lambda < y_\lambda$, then λ is a **multiplicity obstruction**. If also $z_\lambda = 0$, then λ is an **occurrence obstruction**.

Theorem (with Bürgisser and Panova, 2016)

Occurrence obstructions cannot prove superpolynomial lower bounds for (1).

Hope (Mulmuley and Sohoni): For orbit closures of points that are **characterized by their symmetries**: obstructions should be obtainable **from the symmetries**.

Hope for multiplicities

Definition (characterized by stabilizer)

A point p is **characterized by its stabilizer** H, if for every point q with $H \leq \mathrm{stab}_G(q)$ we have $q \in \mathbb{C}p$.

Many points are characterized by their stabilizer: $x_1 \cdots x_m$, $x_1^m + \cdots + x_m^m$, $\det_D$, per_D, $\mathrm{imp}_D^{(m)}$, ...

Proposition

If a point that is characterized by its stabilizer, each of the following pieces of information is sufficient to determine the others:
(1) The orbit Gp (2) The orbit closure $\overline{Gp}$ (3) The stabilizer of p up to conjugation

Proposition

For a polystable point, under technical assumptions (G must be be a compact Lie group and the stabilizer H must be connected and $\mathbb{C}^m$ must be an irreducible H-representation) we can enlarge this list [Yu 2016], based on [Larsen, Pink 1990]:

(4) The multiplicities in the coordinate ring of Gp

(5) The multiplicities in the coordinate ring of $\overline{Gp}$

- $\mathsf{GL}_m p$ is an algebraic variety, $\mathsf{GL}_m p \subseteq \overline{\mathsf{GL}_m p}$ open in its closure.
- $\mathbb{C}[\mathsf{GL}_m p]$: ring of regular functions on the orbit $\mathsf{GL}_m p$. For example $a^4/(b^2 - 4ac) \in \mathbb{C}[\mathsf{GL}_2(x_1)^2]_2$
- $\mathbb{C}[\overline{\mathsf{GL}_m p}] \subseteq \mathbb{C}[\mathsf{GL}_m p]$ subring. Therefore $\mathrm{mult}_\lambda \mathbb{C}[\overline{\mathsf{GL}_m p}] \leq \mathrm{mult}_\lambda \mathbb{C}[\mathsf{GL}_m p]$.

Let $H \leq \mathsf{GL}_m$ be the stabilizer of p.

- For $p = x_1^D + \cdots + x_m^D$ we have $H = C_D \wr \mathfrak{S}_m$
- For $p = x_1 \cdots x_m$ we have $H = \mathsf{ST}_m \rtimes \mathfrak{S}_m$
- For $p = \det_D$, $m = D^2$, we have $H = ((\mathsf{GL}_D \times \mathsf{GL}_D)/\mathbb{C}^\times) \rtimes \mathfrak{S}_2$ (Frobenius 1896).

Algebraic Peter-Weyl theorem

$\mathsf{GL}_m p = \mathsf{GL}_m/H$ via $gp \mapsto gH$. Have a GL_m-action on GL_m/H via $g'(gH) := (g'g)H$.

$\mathbb{C}[\mathsf{GL}_m p] = \mathbb{C}[\mathsf{GL}_m]^H \simeq \bigoplus_\lambda \{\lambda\} \otimes \{\lambda\}^H$. Hence $\mathrm{mult}_\lambda \mathbb{C}[\mathsf{GL}_m p] = \dim(\{\lambda\})^H$.

$\dim(\{\lambda\})^H$ can be calculated using the stabilizer and representation theoretic branching rules:

- For $p = \det_D$ we have $\dim(\{\lambda\})^H = sk(\lambda, D \times \delta)$, a symmetric rectangular Kronecker coefficient
- For $p = x_1 \cdots x_m$ we have $\dim(\{\lambda\})^H = a_\lambda(m, \delta)$, a plethysm coefficient.

Theorem (with Panova), proof via classical representation theoretic branching coefficients

For $p = x_1^D + \cdots + x_m^D$ we have $\dim(\{\lambda\})^H = \sum_{\varrho \vdash_m \delta} b(\lambda, \varrho, D, \delta)$ with

$$b(\lambda, \varrho, D, \delta) := \sum_{\substack{\mu^1,\mu^2,\ldots,\mu^\delta \\ \mu^i \vdash Di\hat{\varrho}_i}} c^\lambda_{\mu^1,\mu^2,\ldots,\mu^\delta} \prod_{i=1}^{\delta} a_{\mu^i}(\hat{\varrho}_i, iD).$$

and $\hat{\varrho}_i = \#\{j \mid i = \varrho_j\}$ (i.e., the frequency notation of ϱ).

$x_0^{p(m)-m}\mathrm{per}_m$ vs $\det_{p(m)}$	occurrence obstructions	multiplicity obstructions
via stabilizer upper bound	No: I-Panova 2015	?
any	No: Bürgisser-I-Panova 2016	?

power sum vs prod of lin forms in few variables	occurrence obstructions	multiplicity obstructions
via stabilizer upper bound	No: "	Yes: Dörfler-I-Panova 2019
any	No: Dörfler-I-Panova 2019	Yes: "

unit tensor vs MaMu tensor	occurrence obstructions	multiplicity obstructions
via stabilizer upper bound	Yes: Bürgisser-I 2011, 2013	Yes: "
any	Yes: "	Yes: "

- All obstructions so far require an evaluation of a function at the "hard problem" (permanent, MaMu, etc).
- This is not intended in the Mulmuley-Sohoni papers.
- [I-Kandasamy 2020] does **not** require an evaluation. We show a tight connection

$$\mathbb{C}[\mathsf{GL}_m(x_1^m + \cdots + x_m^m)] \ \leftrightarrow\ \mathbb{C}[\overline{\mathsf{GL}_m(x_1^m + \cdots + x_m^m)}].$$

Theorem [I, Kandasamy; STOC 2020]

Let $m \geq 3$. Let $Z_m := \overline{\mathsf{GL}_m(x_1 x_2 \cdots x_m)}$. Let $Y_m := \overline{\mathsf{GL}_m(x_1^m + x_2^m + \cdots + x_m^m)}$. Let $\lambda := (4m, \underbrace{2m, 2m, 2m, \ldots, 2m}_{m-1 \text{ many}})$. Then

$$a_\lambda(m+1, m) \geq 3 > \boxed{\mathsf{mult}_\lambda(\mathbb{C}[Y_m]) = 2} > \boxed{1 \geq \mathsf{mult}_\lambda(\mathbb{C}[Z_m])} \overset{m=\text{prime}\pm 1}{>} 0.$$

Therefore

- $Y_m \not\subseteq Z_m$.

and hence $x_1^m + \cdots + x_m^m$ is not a product of homogeneous linear polynomials.

Caveat: Just the non-inclusion on its own is only barely nontrivial as a result.

The bounds are derived from the **symmetry groups** of $x_1^m + \cdots + x_m^m$ and $x_1 \cdots x_m$.

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$$a_\lambda(m+1, m) \geq 3 > \mathsf{mult}_\lambda(\mathbb{C}[Y_m]) = 2 > 1 \geq \mathsf{mult}_\lambda(\mathbb{C}[Z_m]) \overset{m=\text{prime}\pm 1}{>} 0.$$

Therefore

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We have $\mathsf{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_m(x_1 \cdots x_m)}]_\delta) \leq \mathsf{mult}_\lambda(\mathbb{C}[\mathsf{GL}_m(x_1 \cdots x_m)]_\delta) = a_\lambda(m, \delta) \leq K(\lambda, m \times \delta) =$ number of semistandard tableaux of shape λ with numbers $1, \ldots, m$, each exactly δ times.

For the λ in the theorem there is only 1 such tableau!

Example ($m = 4$):

1	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4
2	2	2	2	2	2	2	2								
3	3	3	3	3	3	3	3								
4	4	4	4	4	4	4	4								

Polystability: The fundamental invariant connects $\mathbb{C}[\overline{Gp}]$ and $\mathbb{C}[Gp]$

Let $G := \mathsf{GL}_m$.

- These points are SL_m-polystable, i.e., their SL_m-orbit is closed (Hilbert-Mumford-Luna-Kempf):
  - $x_1 \cdots x_m$
  - $x_1^m + \cdots + x_m^m$
  - $\det_D$ for $m = D^2$
  - per_D for $m = D^2$

Theorem (Bürgisser-I)

If p is polystable, then

- there is an SL_m-invariant $\Phi \in \mathbb{C}[\overline{Gp}]$ of smallest degree.
- Φ cuts out the boundary $\overline{Gp} \setminus Gp \subseteq \overline{Gp}$.
- The ring $\mathbb{C}[Gp]$ is the localization $\mathbb{C}[\overline{Gp}]_\Phi$.

Φ is called the **fundamental invariant**.

- We know Φ for $x_1^D + \cdots + x_m^D$ for even D. For odd D we need $\binom{2D}{D} \geq 2m$.
- For $x_1 \cdots x_m$ we know it for all cases in which the Alon-Tarsi conjecture (Latin squares) is true.

Theorem [I, Kandasamy; STOC 2020]

Let $m \geq 3$. Let $Z_m := \overline{\mathsf{GL}_m(x_1 x_2 \cdots x_m)}$. Let $Y_m := \overline{\mathsf{GL}_m(x_1^m + x_2^m + \cdots + x_m^m)}$. Let $\lambda := (4m, \underbrace{2m, 2m, 2m, \ldots, 2m}_{m-1 \text{ many}})$. Then

$$a_\lambda(m+1, m) \geq 3 > \text{mult}_\lambda(\mathbb{C}[Y_m]) = 2 > 1 \geq \text{mult}_\lambda(\mathbb{C}[Z_m]) \overset{m=\text{prime}\pm 1}{>} 0.$$

Let $p = x_1^m + \cdots + x_m^m$ so $\overline{Gp} = Y_m$.

$\text{mult}_\lambda \mathbb{C}[Gp] = \sum_{\varrho \vdash_m \delta} b(\lambda, \varrho, D, \delta) = 2$, but that is only an **upper** bound for $\text{mult}_\lambda \mathbb{C}[\overline{Gp}]$.

Main Technical Theorem

- If D is even, let $e_\varrho := \sum_{i=1}^m \lceil \frac{\varrho_i}{D-2} \rceil$. If D is odd, let $e_\varrho := \sum_{i=1}^m 2\lceil \frac{\varrho_i}{2(D-2)} \rceil$.
- Let $\lambda \vdash_m \delta D$.
- Let k be the number of $m \times m$ blocks in λ.
- Let $d := \delta - mk$.
- Let $e := \max\{e_\varrho \mid \varrho \vdash_m d\}$.

If $k \geq e$, then $\text{mult}_\lambda \mathbb{C}[Gp] = \text{mult}_\lambda \mathbb{C}[\overline{Gp}]$.

- $\lambda = (4m, 2m, 2m, \ldots, 2m)$
- $k = 2, \qquad D = m, \qquad \delta = 2m^2 + 2m$
- $d = \delta - mk = 2m^2$
- $e = \max\{e_{(2)}, e_{(1,1)}\} = \max\{1, 2\} = 2.$

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A vector f in a GL_m-representation $\mathscr{V}$ is called a **highest weight vector** (HWV) of weight λ, if

- $\text{diag}(\alpha_1, \ldots, \alpha_m) f = \alpha_1^{\lambda_1} \cdots \alpha_m^{\lambda_m} f$
- $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} f = f$

The highest weight vectors of weight λ form a linear subspace $\text{HWV}_\lambda(\mathscr{V}) \subseteq \mathscr{V}$.

- $\text{mult}_\lambda(\mathscr{V}) = \dim \text{HWV}_\lambda(\mathscr{V})$

HWVs can be understood in terms of tableaux

A **Young tableau** or just **tableau** $T : \lambda \to \mathbb{N}$ is an assignment of numbers to the boxes of λ.

$$\begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & & \\ 3 & 4 & 4 & & \\ 4 & & & & \end{array}$$

The **content** of a tableau T is the list: (number of 1s in T, number of 2s in T, $\ldots$).

A Young tableau is called **semistandard** if each column is increasing from top to bottom and every row is nondecreasing from left to right.

The **superstandard tableau** $\mathbb{S}_\lambda$ of shape λ has only entries i in row i:

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & & \\ 3 & 3 & 3 & & \\ 4 & & & & \end{array}$$

Quotient out by the vector space of shuffle relations (Grassmann-Plücker relations)

In the vector space of formal linear combinations of Young tableaux there is the linear subspace $K(\lambda)$, which is spanned by

- $T + T'$, where T' is a tableau that arises from T by switching two numbers within one column.
- $T - \Sigma S$, where for two fixed columns j, j' and a fixed number of entries k the sum is over all tableaux S that arise from T by exchanging the top k entries in column j with any k entries in column j', preserving the internal vertical order.

The action on the space of linear combinations of tableax is defined via multilinear expansion.

Example: $$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1{+}2 & 1{+}2 \\ \hline 2 & 2 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & 2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & 2 \\ \hline \end{array} = 2 \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}.$$

Averaging

- Let $T : \lambda \to \{1, \ldots, \delta\}$ be a tableau.
- Let $\varphi : \{1, \ldots, \delta\} \to \{1, \ldots, m\}$ be a map.
- Define $\varphi T := \varphi(T)$ naively.

Example: $\begin{bmatrix} \varphi(1) = 2 \\ \varphi(2) = 2 \\ \varphi(3) = 1 \end{bmatrix}$ implies $\varphi\left(\begin{matrix}1&1&1&2&3\\2&2&3\\3\end{matrix}\right) = \underbrace{\begin{matrix}2&2&2&2&1\\2&2&1\\1\end{matrix}}_{=0 \text{ in } \{(5,3,1)\}}$.

The "orbit average": $$M_{\delta,m}T := \sum_{\varphi:\{1,\ldots,\delta\}\to\{1,\ldots,m\}} \varphi T$$

Example: $$M_{3,2}\begin{matrix}1&1&1&2&3\\2&2&3\\3\end{matrix} = \begin{matrix}1&1&1&2&1\\2&2&1\\1\end{matrix} + \begin{matrix}1&1&1&2&2\\2&2&2\\2\end{matrix} + \begin{matrix}2&2&2&2&1\\2&2&1\\1\end{matrix} + \begin{matrix}2&2&2&2&2\\2&2&2\\2\end{matrix} + \begin{matrix}1&1&1&1&1\\1&1&1\\1\end{matrix} + \begin{matrix}1&1&1&1&2\\1&1&2\\2\end{matrix} + \begin{matrix}2&2&2&1&1\\1&1&1\\1\end{matrix} + \begin{matrix}2&2&2&1&2\\1&1&2\\2\end{matrix} = 0$$

The symmetrization: $$P_m T := \sum_{\pi \in \mathfrak{S}_m} \pi T$$

Example: $$P_3\begin{matrix}1&1&1&2&3\\2&2&3\\3\end{matrix} = \begin{matrix}1&1&1&2&3\\2&2&3\\3\end{matrix} + \begin{matrix}2&2&2&1&3\\1&1&3\\3\end{matrix} + \begin{matrix}3&3&3&2&1\\2&2&1\\1\end{matrix} + \begin{matrix}1&1&1&3&2\\3&3&2\\2\end{matrix} + \begin{matrix}2&2&2&3&1\\3&3&1\\1\end{matrix} + \begin{matrix}3&3&3&1&2\\1&1&2\\2\end{matrix} = 0$$

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- Let $\lambda \vdash_m \delta D$.
- Let k be the number of $m \times m$ blocks in λ.
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If $k \geq e$, then $\text{mult}_\lambda \mathbb{C}[Gp] = \text{mult}_\lambda \mathbb{C}[\overline{Gp}]$.

Theorem (HWVs in the coordinate ring of the orbit closure)

The vector space $\text{HWV}_\lambda(\mathbb{C}[\overline{Gp}]_\delta)$ is generated by the functions

$$g \mapsto \langle \mathbb{S}_\lambda, g M_{\delta,m} T \rangle \,,$$

where T runs over all semistandard tableaux of shape λ in which each entry $1, \ldots, \delta$ appears exactly D times.

Theorem (HWVs in the coordinate ring of the orbit)

The vector space $\text{HWV}_\lambda(\mathbb{C}[Gp]_\delta)$ decomposes into a direct sum of vector spaces $\text{HWV}_\lambda(\mathbb{C}[Gp]_\delta) = \bigoplus_{\varrho \vdash_m \delta} \mathscr{W}_\varrho$, and each $\mathscr{W}_\varrho$ is generated by the functions

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where S runs over all semistandard tableaux S of shape λ and content ϱD.

Example: $D = 8,\ \lambda = (24, 8, 8, 8, 8, 8, 8, 8),$

$$S = \begin{array}{cccccccccccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
2&2&2&2&2&2&2&2\\
3&3&3&3&3&3&3&3\\
4&4&4&4&4&4&4&4\\
5&5&5&5&5&5&5&5\\
6&6&6&6&6&6&6&6\\
7&7&7&7&7&7&7&7\\
8&8&8&8&8&8&8&8
\end{array}
\qquad \varrho = (3, 1, 1, 1, 1, 1, 1, 1)$$

$$P_8 S = 7! \left(\begin{array}{cccccccccccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
2&2&2&2&2&2&2&2\\
3&3&3&3&3&3&3&3\\
4&4&4&4&4&4&4&4\\
5&5&5&5&5&5&5&5\\
6&6&6&6&6&6&6&6\\
7&7&7&7&7&7&7&7\\
8&8&8&8&8&8&8&8
\end{array}
+ \cdots +
\begin{array}{cccccccccccccccccccccccc}
1&1&1&1&1&1&1&1&8&8&8&8&8&8&8&8&8&8&8&8&8&8&8&8\\
2&2&2&2&2&2&2&2\\
3&3&3&3&3&3&3&3\\
4&4&4&4&4&4&4&4\\
5&5&5&5&5&5&5&5\\
6&6&6&6&6&6&6&6\\
7&7&7&7&7&7&7&7\\
8&8&8&8&8&8&8&8
\end{array} \right)$$

From

$S =$

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2																
3	3	3	3	3	3	3	3																
4	4	4	4	4	4	4	4																
5	5	5	5	5	5	5	5																
6	6	6	6	6	6	6	6																
7	7	7	7	7	7	7	7																
8	8	8	8	8	8	8	8																

we construct

$T =$

1	1	2	2	3	3	3	3	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3
4	4	4	4	4	4	4	4																
5	5	5	5	5	5	5	5																
6	6	6	6	6	6	6	6																
7	7	7	7	7	7	7	7																
8	8	8	8	8	8	8	8																
9	9	9	9	9	9	9	9																
10	10	10	10	10	10	10	10																

$M_{10,8}T = 7! \Bigg($

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2																
3	3	3	3	3	3	3	3																
4	4	4	4	4	4	4	4																
5	5	5	5	5	5	5	5																
6	6	6	6	6	6	6	6																
7	7	7	7	7	7	7	7																
8	8	8	8	8	8	8	8																

$+ \cdots +$

1	1	1	1	1	1	1	1	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
2	2	2	2	2	2	2	2																
3	3	3	3	3	3	3	3																
4	4	4	4	4	4	4	4																
5	5	5	5	5	5	5	5																
6	6	6	6	6	6	6	6																
7	7	7	7	7	7	7	7																
8	8	8	8	8	8	8	8																

$\Bigg) = P_8 S$

Proof of the technical theorem: This works for all λ with enough full $m \times m$ blocks.

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$$a_\lambda(m+1, m) \geq 3 > \boxed{\mathsf{mult}_\lambda(\mathbb{C}[Y_m]) = 2} > 1 \geq \mathsf{mult}_\lambda(\mathbb{C}[Z_m]) \overset{m=\text{prime}\pm 1}{>} 0.$$

Summary

- The power sum does not factor as a product of linear forms.
- This is shown via explicit multiplicity obstructions $\lambda = (4m, 2m, 2m, \ldots, 2m)$.
- Not occurrence obstructions (in infinitely many cases).
- Not vanishing ideal occurrence obstructions.
- **Both** multiplicities are obtained via representation theoretic branching formulas.
- This is achieved by proving a close connection between $\mathbb{C}[Gp]$ and $\mathbb{C}[\overline{Gp}]$, $p = x_1^m + \cdots + x_m^m$.

Thank you for your attention!