How to prove Orbits are Closed?

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Polystability in positive char....

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<u>Motivation</u>: Deg. lower bounds for invariant rings

Previous talk: char 0

$SL_n \times SL_n \times SL_n \curvearrowright (\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n)^{\oplus 5}$

$SL_n \curvearrowright (\mathrm{Sym}^3 \mathbb{C}^n)^{\oplus 4}$

exp. deg. l. bds.

very unhappy w/ char 0

<u>Grosshans principle</u>:

$G \curvearrowright V, W$

$v \in V$ closed orbit

$\mathrm{Stab}_G(v) = H$

$\mathbb{C}[V \oplus W]^G \twoheadrightarrow \mathbb{C}[W]^H$

deg. bd $\geq$ deg. bd

In this motivation: Need to (try) to verify whether points with symmetries have closed orbits.

Char 0: Moment map $\rightarrow$ can help you identify closed orbits

My main motivation: Do this without analysis? So that you can do for char $p>0$

$K^{3n} \longrightarrow$ basis $\{x_i, y_i, z_i\}$, $1 \le i \le n$

Eg: $V = \left(\sum x_i^2 z_i, \ \sum y_i^2 z_i\right) \in \mathrm{Sym}^3 (K^{3n})^{\oplus 2}$

Example:

$$h_3(x,y,z) = x^3 + y^3 + z^3 + x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 + xyz.$$

Qn: Is the SL_3 orbit of $h_3(x,y,z)$ closed?

Does it depend on characteristic?

$$\boxed{\mathrm{char}(K) = p}$$

How does it depend on characteristic?

Note: Moment map criterion $\rightarrow$ doesn't work for $h_3(x,y,z)$

$SL_3 \curvearrowright \mathrm{Sym}^3(K^3) \to$ deg 3 polys in $\{x, y, z\}$

$K^3 \to$ basis x, y, z.

Eg: $f = x^3 + x^2 y + z^3$

$$\lambda(t) = \begin{bmatrix} t & & \\ & t^{-1} & \\ & & 1 \end{bmatrix} \in SL_3$$

$$\lambda(t).\ x = tx$$
$$y = t^{-1} y$$
$$z = z$$

$$\lambda(t) \cdot f = (tx)^3 + (tx)^2 (t^{-1} y) + z^3$$
$$= t^3 x^3 + t x^2 y + z^3$$

$$\lim_{t \to 0} \lambda(t) \cdot f = z^3 \in \overline{SL_n \cdot f}$$

Claim: $z^3 \notin SL_n \cdot f$.

Hilbert-Mumford Criterion:

$G\cdot v$ is not closed

$$\Updownarrow$$

$\exists$ 1-parameter subgroup $\lambda : K^* \longrightarrow G$

s.t $\lim_{t\to 0} \lambda(t)\cdot v \notin G\cdot v$

$G = SL_n$

Any 1-psg is of the form

$$\lambda(t) = g \begin{bmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_n} \end{bmatrix} g^{-1}$$

for some $g \in SL_n$.

Want to show $G\cdot v$ not closed $\Rightarrow$ show (by magic) an appropriate $\lambda(t)$

Want to show $G\cdot v$ closed $\Rightarrow$ argue over all 1-psg's.

Kempf: Instability in invariant theory.

Suppose $G \cdot v$ not closed.

Then

1. $\exists!$ optimal parabolic subgroup $P_{G,v}$
2. For any max'l torus $T \subseteq P_{G,v}$, $\exists$ an optimal 1-psg $\lambda(t) \in T$ that drives v out of the orbit.
3. $\mathrm{Stab}_G(v) \subseteq P_{G,v}$

High-level idea: Search for an optimal 1-psg. If you fail, the orbit is closed. If you succeed, then you verifiably know that the orbit is not closed.

Concrete strategy:

① Collect all $\overbrace{\text{parabolics} \supseteq \mathrm{Stab}_G(v)}^{\mathcal{P}}$

② Find a collection of maximal tori $\mathcal{T}$ so that for for any $P \in \mathcal{P}$, $\exists\ T \in \mathcal{T}$ s.t $T \subseteq P$

③ For each $T \in \mathcal{T}$, check if $\overline{T.v} \subseteq G.v$ or not.

Catch: Hopefully $\mathcal{T}$ is manageable.

Key: $\mathrm{Stab}_G(v)$ non-trivial

Parabolic Subgps of SL_n $\longleftrightarrow$ Flags in K^n.

$$P_{\mathcal{F}} \longleftrightarrow \mathcal{F}$$

$\mathcal{F} = \{ 0 = F_0 \subseteq F_1 \subset \dots \subseteq F_d = K^n \}$

F_i lin. subspaces

$\mathrm{Stab}_G(v) \subseteq P_{\mathcal{F}} \iff$ each F_i are $\mathrm{Stab}_G(v)$ - stable

Parabolics $\supseteq \underset{\substack{\| \\ H}}{\mathrm{Stab}_G(v)} \longleftrightarrow$ Flags of $H = \mathrm{Stab}_G(v)$ - stable subspaces

Eg: $h_3(x,y,z)$ $G = SL_3$.

$H = \mathrm{Stab}_G(h_3(x,y,z)) \supseteq S_3$

Parabolic containing $S_3 \longleftrightarrow$ Flags of S_3-stable subspaces of K^3

S_3-stable subspaces of K^3 $=$ $\{0\}$

$\langle x+y+z \rangle = L$

$\langle x-y, y-z \rangle = M$

$\shortparallel$

$\{ c_1 x + c_2 y + c_3 z \mid \sum c_i = 0 \}$

K^3

Possible flags:

$$\boxed{\begin{array}{l} 0 \subseteq L \subseteq K^3 \\ 0 \subseteq M \subseteq K^3 \\ \hline 0 \subseteq K^3 \end{array}}$$

$0 \subseteq L \subseteq M \subseteq K^3 \rightarrow$ only in char 3

Maximal tori of SL_n

Std one $\longrightarrow ST_n = \left\{ \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \middle| \prod d_i = 1 \right\}$

Maximal tori $\longleftrightarrow$ Bases $(b_1, \dots, b_n)$

$T_B = B(ST_n)B^{-1} \longleftrightarrow B$

What does it mean for

$T_B \subseteq P_{\mathcal{F}_\bullet}$?

$\{0\} = F_0 \subseteq F_1 \subset \dots \subseteq F_d = K^n\}$

Each F_i is a <u>coordinate subspace</u> in the basis B.

F_i = span of some subset of $b_1 - b_n$

Take a basis for F_1, extend to F_2 & so on

Eg: $h_3(x,y,z)$

① $0 \subseteq L \subseteq K^3$ $\qquad$ $L = \langle x+y+z \rangle$

② $0 \subseteq M \subseteq K^3$ $\qquad$ $M = \langle x-y,\ y-z \rangle$

Fact 1: Nothing non-trivial comes out of ①

So I'll just show ②.

$0 \subseteq \underset{\substack{\shortparallel \\ P}}{M} \subseteq K^3 \rightsquigarrow$ bases.

$B = (x-y,\ y-z,\ z)$

Check: $T_B \subseteq P$.

Need to understand $T_B \cdot h_3(x,y,z)$

& $\overline{T_B \cdot h_3(x,y,z)}$

$T_B \cdot h_3(x,y,z)$

$\shortparallel$

$B(ST_n)B^{-1}$

$\downarrow$

Write $h_3(x,y,z)$ in the basis B

& then work with ST_n

$$B = (x-y,\ y-z,\ z)$$
$$\quad\ a \qquad b \qquad c$$

$$x = a+b+c$$
$$y = b+c$$
$$z = c$$

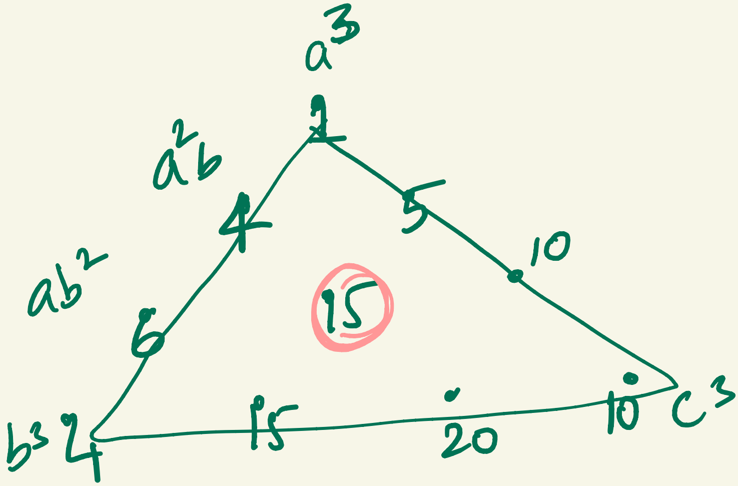
$$h_3(x,y,z) = h_3(a+b+c,\ b+c,\ c).$$

$h_3(a+b+c,\ b+c,\ c)$

$$= a^3 + 4a^2b + 6ab^2 + 4b^3 + 5a^2c + 10ac^2 + 10c^3 + 15b^2c + 20bc^2 + 15abc.$$

$\downarrow$

Plot the newton polytope



char 0 $\longrightarrow$ Newton polytope contains 0

$\Rightarrow J_B \cdot h_3(x,y,z)$ is closed.

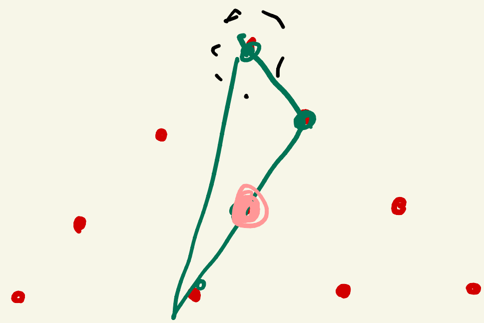
If $p \neq 2, 5$

(Newton polytope contains origin
∴ along w/ analogous computation for other parabolic)

$\Downarrow$

SL_3 orbit of $h_3(x,y,z)$ is closed.

$p = 2$



Origin on the boundary of the newton polytope

$$\lambda(t) = \begin{bmatrix} t & & \\ & t^{-2} & \\ & & t \end{bmatrix}$$

$$\lambda(t).\, a = ta$$
$$b = tb$$
$$c = t^{-2}c$$

$h_3(x,y,z)$

$h_3(a+b+c,\ b+c,\ c)$

$\|$

$a^3 + a^2c + abc + b^2c$

$\downarrow \lambda(t)$, $\lim t \to \infty$

$t^3a^3 + a^2c + abc + b^2c.$

$\| $

$c(a^2+ab+b^2).$

$$c(a^2+ab+b^2) = \lim_{t\to 0} \lambda(t)\, h_3(x,y,z)$$

$\Uparrow$

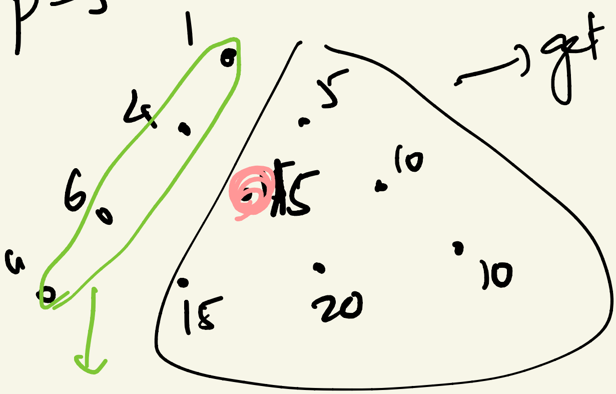
Orbit closure.

Eisenstein criterion $\Downarrow$

$h_3(x,y,z)$ is irreducible.

Cannot be in the orbit b/c it is reducible.

$p=5$



get killed

Newton polytope.

Origin outside newton polytope

$\Downarrow$

$h_3(x,y,z)$ is in the null cone.

$\downarrow$

$$0 \in \overline{SL_3 \cdot h_3(x,y,z)}$$

Conclusion:

$h_3(x, y, z)$

SL_3 orbit

- closed if $p \neq 2, 5$
- not closed but not in the null cone if $p = 2$
- in the null cone if $p = 5$.

→ Schur polynomials → have closed orbits even in char 0

→ Algorithm to decide if a symmetric polynomial has a closed orbit.