Highest weight vectors, power sums, and Young tableaux
(Implementing GCT: On the separation of orbit closures via symmetries)

Christian Ikenmeyer
Let $G := GL_m$, $p = x_1^D + \cdots + x_m^D$.
We have formulas for $\text{mult}_\lambda(\mathbb{C}[Gp])$.
We want formulas for $\text{mult}_\lambda(\mathbb{C}[\overline{Gp}])$.

We find a connection between these two via tableau-theoretic interpretations of both multiplicities.

This leads to the first multiplicity obstructions for orbit closure containment, based only on symmetry groups (2nd talk).
(with Kandasamy, STOC 2020)

This talk:
A tableau-theoretic interpretation of $\text{mult}_\lambda(\mathbb{C}[\overline{Gp}])$. 
1. Representation theoretic multiplicities and highest weight polynomials

2. This talk’s main theorem: Explicit construction of highest weight polynomials for the power sum via Young tableaux

3. Proof sketch via multilinear algebra
1. Representation theoretic multiplicities and highest weight polynomials

2. This talk’s main theorem: Explicit construction of highest weight polynomials for the power sum via Young tableaux

3. Proof sketch via multilinear algebra
Let $\text{Sym}^D C^m := C[x_1, \ldots, x_m]_D$.

Let $G := \text{GL}_m$

$G$ acts linearly on $\text{Sym}^D C^m$ via $(gp)(x) := p(g^{-1}x)$.

Let $X \subseteq \text{Sym}^D C^m$ be a closed subvariety that is closed under the action of $G$.

For example, $X = \overline{Gp}$ an orbit closure.

Let $\text{Sym}^\delta \text{Sym}^D C^m := C[\text{Sym}^D C^m]_\delta$.

$G$ acts linearly on $\text{Sym}^\delta \text{Sym}^D C^m$ via $(gf)(p) := f(g^{-1}p)$, i.e., $\text{Sym}^\delta \text{Sym}^D C^m$ is a $G$-representation.

Let $I(X) = \{ f \in C[\text{Sym}^D C^m] \mid f(X) = \{0\} \}$ be the vanishing ideal.

Let $I(X)_\delta = \{ f \in \text{Sym}^\delta \text{Sym}^D C^m \mid f(X) = \{0\} \}$ be the homogeneous degree $\delta$ component of the vanishing ideal.

Since $X$ is closed under the action of $G$, $I(X)_\delta$ is also closed under the action of $G$, i.e., $I(X)_\delta$ is a $G$-representation.

$C[X] = C[\text{Sym}^D C^m]/I(X)$ is the coordinate ring.

Alternatively: $C[X] \simeq \{ f|_X \mid f \in \text{Sym}^D C^m \}$ via restrictions of the domain of definition to $X$.

$C[X]_\delta = \text{Sym}^\delta \text{Sym}^D C^m/I(X)_\delta$.

Since $\text{Sym}^\delta \text{Sym}^D C^m$ is a $G$-representation and $I(X)_\delta$ is a subrepresentation, it follows that $C[X]_\delta$ is a $G$-representation.
Young tableaux

- A **partition** is finite nonincreasing list of natural numbers. E.g., $\lambda = (5, 3, 3, 1)$
- The top-left-justified array of boxes with $\lambda_i$ boxes in row $i$ is called the **Young diagram** of $\lambda$.

![Young diagram of $\lambda$]

The transpose $\lambda^t$:

- We often identify $\lambda$ with its Young diagram.

We write $\lambda \vdash m \ d$ if $\lambda$ is a Young diagram with $d$ boxes and at most $m$ rows.
- For $\lambda \vdash d$, we write $[\lambda]$ for the irreducible $S_d$-representation of type $\lambda$.
- For $\lambda \vdash m$, we write $\{\lambda\}$ for the irreducible $GL_m$-representation of type $\lambda$.
- A **Young tableau** or just **tableau** $T : \lambda \to \mathbb{N}$ is an assignment of numbers to the boxes of $\lambda$.

![Young tableau]

- A Young tableau is called **semistandard** if each column is increasing from top to bottom and every row is nondecreasing from left to right.

The **superstandard tableau** $S_{\lambda}$ of shape $\lambda$ has only entries $i$ in row $i$:

![Superstandard tableau]

The **column-standard tableau** $T_{\lambda}$ of shape $\lambda$:
Multiplicities and highest weight polynomials

- Every $GL_m$-representation $\mathcal{V}$ decomposes (not necessarily uniquely) into a direct sum of irreducible subrepresentations:

$$\mathcal{V} = \bigoplus_{\lambda \vdash m} \{\lambda\}^{c_\lambda}$$

For example:

$$\text{Sym}^\delta \text{Sym}^D \mathbb{C}^m = \bigoplus_{\lambda \vdash m} \{\lambda\}^{a_\lambda(\delta, D)}$$

- These nonnegative integers $c_\lambda$ (and hence also $a_\lambda(\delta, D)$) are independent of the decomposition. They are called representation theoretic multiplicities. $\text{mult}_\lambda(\mathcal{V}) := c_\lambda$. $a_\lambda(\delta, D)$ is called a plethysm coefficient.

Every irreducible $GL_m$-representation contains exactly one (up to scale) highest weight vector (HWV):

- $\text{diag}(\alpha_1, \ldots, \alpha_m) f = \alpha_1^{\lambda_1} \cdots \alpha_m^{\lambda_m} f$

- $$\begin{pmatrix} 1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} f = f$$

The exponent vector $\lambda$ is called the weight of $f$.

Two irreducible $GL_m$-representations are isomorphic iff the weights of their HWVs coincide.

Example: $g(b^2 - 4ac) = \det(g)^2 (b^2 - 4ac)$, hence $b^2 - 4ac$ is a HWV of weight $(2, 2)$.

- $\text{mult}_\lambda(\mathcal{V}) = \dim \text{HWV}_\lambda(\mathcal{V})$

- $\text{mult}_\lambda(\mathbb{C}[X]_\delta) = \dim \left( \left( \text{HWV}_\lambda(\text{Sym}^\delta \text{Sym}^D \mathbb{C}^m) \right) | X \right)$.
Some papers with coauthors where HWVs are explicitly constructed from Young tableaux:

1. **16,051 formulas for Ottaviani’s invariant of cubic threefolds**, with Abdesselam and Royle 2014: 
   \( \lambda = (9, 9, 9, 9) \), \( D = 3 \), \( \delta = 15 \).

2. **Symmetrizing Tableaux and the 5th case of the Foulkes Conjecture**, with Cheung and Mkrtchyan 2015: 
   \( (\delta, D) = (5, 5), (5, 6), (6, 6) \).

3. **Fundamental invariants of orbit closures**, with Bürgisser 2015:
   rectangular \( \lambda \).

4. **On GCT: Multiplicity obstructions are stronger than occurrence obstructions**, with Dörfler and Panova 2019: 
   \( D = 6, 7 \).

5. **On the complexity of evaluating highest weight vectors**, with Bläser and Dörfler 2020:
   HWV evaluation is NP-hard. Efficient algorithms for evaluation for low tree-width.

6. **Equations for GL invariant families of polynomials** with Breiding, Michałek, Hodges 2021:
   \( D = 3, \delta = 11, \lambda = (15, 6, 6, 6) \). Software and database.
   Uses Hodges’ fast tableau straightening code, arXiv:1710.05214

7. **Implementing GCT: On the separation of orbit closures via symmetries**, with Kandasamy 2019:
   HWV construction is implicit from the existence of certain invariants.

Many other authors, because this is a very classical approach.
In this talk we study the power sum:

- $p := x_1^D + \cdots + x_m^D \in \text{Sym}^D \mathbb{C}^m$.
- $G := \text{GL}_m$.

We want to understand $\text{HWV}_\lambda(\mathbb{C}[G_p]_\delta)$ in terms of Young tableaux.
1 Representation theoretic multiplicities and highest weight polynomials

2 This talk’s main theorem: Explicit construction of highest weight polynomials for the power sum via Young tableaux

3 Proof sketch via multilinear algebra
T(r,c): entry of T in row r and column c.

Let \{e_i\} \subset \mathbb{C}^m standard basis. Let \( \mu = \lambda^t \).

Tableau \( T : \lambda \to \{1, \ldots, m\} \)

\( \uparrow \) 1:1 correspondence

Basis vector \( e_{T(1,1)} \otimes e_{T(2,1)} \otimes \cdots \otimes e_{T(\mu_1,1)} \otimes e_{T(1,2)} \otimes e_{T(2,2)} \otimes \cdots \otimes e_{T(\mu_2,2)} \otimes \cdots \otimes e_{T(\mu_\lambda_1,\lambda_1)} \in \otimes |\lambda| \mathbb{C}^m \)

This gives a vector space isomorphism: \{linear combinations of tableaux of shape \( \lambda \}\} \sim \otimes |\lambda| \mathbb{C}^m.

The \( G \)-action, which is given by \( g(v_1 \otimes \cdots \otimes v_{|\lambda|}) = g(v_1) \otimes \cdots \otimes g(v_{|\lambda|}) \), is inherited:

Example:

\[
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.
\]

The vector space of shuffle relations (Grassmann-Plücker relations)

The subrepresentation \( K(\lambda) \) is spanned by

- \( T + T' \), where \( T' \) is a tableau that arises from \( T \) by switching two numbers within one column.
- \( T - \Sigma S \), where for two fixed columns \( j, j' \) and a fixed number of entries \( k \) the sum is over all tableaux \( S \) that arise from \( T \) by exchanging the top \( k \) entries in column \( j \) with any \( k \) entries in column \( j' \), preserving the internal vertical order.

Theorem

\( \{\lambda\} \simeq (\otimes |\lambda| \mathbb{C}^m)/K(\lambda) \). Basis: the semistd tableaux of shape \( \lambda \) with entries from \{1, \ldots, m\}. HWV: \( S_\lambda \)

\[
\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 4 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix}.
\]

If we insist that every 1, \ldots, m appears exactly once, we obtain the Specht modules \([\lambda]\), i.e., the irreps of \( S_m \).
Averaging

- Let $T: \lambda \to \{1, \ldots, \delta\}$ be a tableau.
- Let $\varphi: \{1, \ldots, \delta\} \to \{1, \ldots, m\}$ be a map.
- Define $\varphi T := \varphi(T)$ naively.

Example:

\[
\begin{bmatrix}
\varphi(1) = 2 \\
\varphi(2) = 2 \\
\varphi(3) = 1
\end{bmatrix}
\text{ implies } \varphi \left(\begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\
2 & 2 & 3 \\
3 \end{bmatrix}\right) = \left(\begin{bmatrix} 2 & 2 & 2 & 2 & 1 \\
2 & 2 & 1 \\
1 \end{bmatrix}\right).
\]

The “orbit average”:

\[
M_{\delta,m}T := \sum_{\varphi: \{1, \ldots, \delta\} \to \{1, \ldots, m\}} \varphi T \in \{\lambda\}^{S_m}
\]

Example: $M_{3,2,1} = 0$ in $\{(5,3,1)\}$

Main Theorem of this talk (coordinate ring of the orbit closure of the power sum expressed via tableaux)

Let $p := x_1^D + \cdots + x_m^D \in \text{Sym}^D \mathbb{C}^m$. Let $G := \text{GL}_m$.

Let $\lambda \vdash \delta D$. The vector space $\text{HWV}_\lambda(\mathbb{C}[Gp]_\delta)$ is generated by the functions

\[
g \mapsto \langle S_\lambda, gM_{\delta,m}T \rangle, \quad \langle S, T \rangle = \delta_{S,T}
\]

where $T$ runs over all semistandard tableaux of shape $\lambda$ in which each entry $1, \ldots, \delta$ appears exactly $D$ times.
\[ M_{\delta,m}T := \sum_{\varphi : \{1, \ldots, \delta\} \rightarrow \{1, \ldots, m\}} \varphi(T) \]

Let \( p := x_1^D + \cdots + x_m^D \in \text{Sym}^D \mathbb{C}^m \). Let \( G := \text{GL}_m \). Let \( \lambda \vdash \delta D \).

**Main Theorem of this talk (coordinate ring of the orbit closure of the power sum expressed via tableaux)**

The vector space \( \text{HWV}_\lambda(\mathbb{C}[Gp]_\delta) \) is generated by the functions

\[
g \mapsto \langle S_\lambda, gM_{\delta,m}T \rangle, \quad \langle S, T \rangle = \delta_{S,T} \]

where \( T \) runs over all semistandard tableaux of shape \( \lambda \) in which each entry \( 1, \ldots, \delta \) appears exactly \( D \) times.

**Well-definedness:**

\( \text{stab}(p) \simeq C_D \wr \mathfrak{S}_m \)

For \( q \in Gp \) we have \( q = gp \), but \( g \) is unique only up to right-mult. with \( \text{stab}(p) \): \( gp = ghp \) with \( h \in \text{stab}(p) \).

**Example** \((D = 3, \delta = 4, m = 2)\)

The cyclic group part: If \( \omega^3 = 1 \), then \( \text{diag}(\omega, 1) \begin{array}{c} 1 \ 1 \ 1 \\ 2 \ 2 \ 2 \ 1 \end{array} = \begin{array}{c} 1 \ 1 \ 2 \ 1 \ 1 \ 1 \\ 2 \ 2 \ 1 \end{array} \)

The symmetric group part: If \( h \in \mathfrak{S}_m \), then \( hM_{\delta,m}T \) and \( M_{\delta,m}T \) are the same sum, just permuted order of summands.
Main Theorem of this talk (coordinate ring of the orbit closure of the power sum expressed via tableaux)

Let \( p := x_1^D + \cdots + x_m^D \in \text{Sym}^D \mathbb{C}^m \). Let \( G := \text{GL}_m \).

Let \( \lambda \vdash \delta D \). The vector space \( \text{HWV}_\lambda(\mathbb{C}[Gp],\delta) \) is generated by the functions

\[
S_\lambda, g M_\delta, m T \mapsto \langle S_\lambda, (g M_\delta, m T) \rangle,
\]

where \( T \) runs over all semistandard tableaux of shape \( \lambda \) in which each entry \( 1, \ldots, \delta \) appears exactly \( D \) times.

\[
\text{Discriminant } T = \begin{array}{c|c|c|c}
1 & 1 & 1 \\
2 & 2 & 2 \\
\end{array}
\]

Discriminant at \( x_1^2 \) is zero: \( M_{2,1} T = \begin{array}{c|c|c|c}
1 & 1 & 1 \\
4 & 1 & 1 \\
\end{array} = 0 \). It follows that it vanishes on Waring rank 1.

Discriminant at \( x_1^2 + x_2^2 \) is nonzero: \( M_{2,2} T = \begin{array}{c|c|c|c} 
1 & 1 & 2 \\
2 & 2 & 2 \\
\end{array} + \begin{array}{c|c|c|c} 
1 & 1 & 1 \\
2 & 2 & 2 \\
\end{array} + \begin{array}{c|c|c|c} 
2 & 2 & 1 \\
1 & 1 & 1 \\
\end{array} = 2 \begin{array}{c|c|c|c} 
1 & 1 & 1 \\
2 & 2 & 2 \\
\end{array}, \quad \langle \begin{array}{c|c|c|c} 
1 & 1 & 1 & 2 \\
4 & 2 & 2 & 2 \\
\end{array}, \begin{array}{c|c|c|c} 
1 & 1 & 1 \\
2 & 2 & 2 \\
\end{array} \rangle = \frac{1}{2} \neq 0
\]

Aronhold invariant: The HWV to

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
1 & 1 & 1 & 2 \\
2 & 2 & 3 & 3 \\
3 & 4 & 4 & 4 \\
\end{array}
\]

vanishes on Waring rank 3. But one calculates that it does not vanish on \( x_1^3 + x_2^3 + x_3^3 + (x_1 + x_2 + x_3)^3 \in \text{GL}_4(x_1^3 + \cdots + x_4^3) \).
Let $p := x_1^D + \cdots + x_m^D \in \text{Sym}^D \mathbb{C}^m$. Let $G := \text{GL}_m$. Let $\lambda \vdash \delta D$.

### Main Theorem of this talk (coordinate ring of the orbit closure of the power sum expressed via tableaux)

The vector space $\text{HWV}_\lambda(\mathbb{C}[Gp]_\delta)$ is generated by the functions

$$g \mapsto \langle S_\lambda, gM_{\delta,m}T \rangle,$$

where $T$ runs over all semistandard tableaux of shape $\lambda$ in which each entry $1, \ldots, \delta$ appears exactly $D$ times.

In the second talk we will see how to construct such functions, using a similar description for functions in $\mathbb{C}[Gp]$:

### Theorem (HWVs in the coordinate ring of the orbit)

The vector space $\text{HWV}_\lambda(\mathbb{C}[Gp]_d)$ decomposes into a direct sum of vector spaces $\text{HWV}_\lambda(\mathbb{C}[Gp]_d) = \bigoplus_{\varrho \vdash m} \mathcal{W}_\varrho$, and each $\mathcal{W}_\varrho$ is generated by the functions

$$g \mapsto \langle S_\lambda, gP_mS \rangle,$$

where $S$ runs over all semistandard tableaux $S$ of shape $\lambda$ and content $\varrho D$. 

Christian Ikenmeyer
1 Representation theoretic multiplicities and highest weight polynomials

2 This talk’s main theorem: Explicit construction of highest weight polynomials for the power sum via Young tableaux

3 Proof sketch via multilinear algebra
\[
\bigotimes_{\delta D} \mathbb{C}^m \rightarrow \Sym^\delta \Sym^D \mathbb{C}^m \rightarrow \bigcup \mathbb{C}[G^\delta]_{\delta}
\]

\[
\text{HWV}_\lambda\big(\bigotimes_{\delta D} \mathbb{C}^m\big) \rightarrow \text{HWV}_\lambda\big(\Sym^\delta \Sym^D \mathbb{C}^m\big) \rightarrow \text{HWV}_\lambda\big(\mathbb{C}[G^\delta]\big)
\]

**Theorem (Schur-Weyl duality)**

\[
\bigotimes_{\delta D} \mathbb{C}^m = \bigoplus_{\lambda \vdash m} \{\lambda\} \otimes [\lambda], \quad \text{HWV}_\nu\big(\bigotimes_{\delta D} \mathbb{C}^m\big) \cong [\nu]
\]

This implies: If we find \(v_\lambda \in \text{HWV}_\lambda\big(\bigotimes_{\delta D} \mathbb{C}^m\big)\), then

\[
\text{HWV}_\lambda\big(\bigotimes_{\delta D} \mathbb{C}^m\big) = \text{linspan}\{\pi v_\lambda \mid \pi \in \mathcal{S}_{\delta D}\}.
\]

An example of such a \(v_\lambda \in \text{HWV}_\lambda\big(\bigotimes_{\delta D} \mathbb{C}^m\big)\):

\[
v_\lambda := e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_1} \otimes e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_2} \otimes \cdots \otimes e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_}\lambda_1,
\]

where \(\mu = \lambda^t\).

Example: \(e_1 \wedge e_2 \otimes e_1 \wedge e_2 = \frac{1}{2} (e_1 \otimes e_2 - e_2 \otimes e_1) \otimes \frac{1}{2} (e_1 \otimes e_2 - e_2 \otimes e_1)\)

\[
= \frac{1}{4} (e_1 \otimes e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_2 \otimes e_1) \in \text{HWV}_{(2,2)}\big(\bigotimes^4 \mathbb{C}^2\big)
\]

\[
\downarrow \quad (23)
\]

\[
\frac{1}{4} (e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_2 \otimes e_1 \otimes e_1) \in \text{HWV}_{(2,2)}\big(\bigotimes^4 \mathbb{C}^2\big)
\]
Defining polynomials via tensor contraction

Let  \( v \in \bigotimes^\delta D C^m \).

\[ \forall y \in \text{Sym}^D C^m : f(y) := \langle v, y \otimes \delta \rangle \text{ defines a polynomial } f \in \text{Sym}^\delta \text{Sym}^D C^m. \]

- **Example (evaluation at \( x_1^2 + x_2^2 \), which as a tensor is \( e_1 \otimes e_1 + e_2 \otimes e_2 \))**:
  \[
  \langle \frac{1}{4} (e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_2 \otimes e_1 \otimes e_1), (e_1 \otimes e_1 + e_2 \otimes e_2) \otimes^2 \rangle \\
  = \frac{1}{4} \langle 1122 - 2112 - 1221 + 2211, 1111 + 1122 + 2211 + 2222 \rangle = \frac{1}{4} \cdot 2 = \frac{1}{2}
  
  Evaluation at \( x_1 x_2 \), which as a tensor is \( \frac{1}{2} (e_1 \otimes e_2 + e_2 \otimes e_1) \):
  \[
  \langle \frac{1}{4} (e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_2 \otimes e_1 \otimes e_1), (\frac{1}{2} (e_1 \otimes e_2 + e_2 \otimes e_1)) \otimes^2 \rangle \\
  = \frac{1}{16} \langle 1122 - 2112 - 1221 + 2211, 1212 + 1221 + 2112 + 2121 \rangle = \frac{1}{16} \cdot (-2) = -\frac{1}{8}
  
  Remark that we recognize this: \(-\frac{1}{8} b^2 + \frac{1}{2} ac = -\frac{1}{4} (b^2 - 4ac)\), the discriminant.

- **Alternative way to calculate via keeping the determinants**:
  \[
  \langle (23)(1 \wedge 2 \otimes 1 \wedge 2), 1111 + 1122 + 2211 + 2222 \rangle = \langle 1 \wedge 2 \otimes 1 \wedge 2, 1111 + 1212 + 2121 + 2222 \rangle \\
  = \frac{1}{4} (\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \det \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \det \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}) = \frac{1}{2}
  
  \[
  T = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, S_{2,2} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, M_{2,2}T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \prod_i \mu_i(S_{2,2}, M_{2,2}T) = \frac{1}{4} \cdot 2 = \frac{1}{2}.
  
  Christian Ikenmeyer
Evaluation at the power sum

Let \( p := x_1^D + \cdots + x_m^D \in \text{Sym}^D C^m \).

Let \( gp = \ell_1^D + \cdots + \ell_m^D \), i.e., \( \ell_i = \ell_{g..m,i} \).

Let \( y := gp \).

We calculate

\[
y \otimes \delta = \sum_{\varphi: \{1, \ldots, \delta\} \to \{1, \ldots, m\}} \ell^D_{\varphi(1)} \otimes \cdots \otimes \ell^D_{\varphi(\delta)}.
\]

Recall \( v_{\lambda} := e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_1} \otimes e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_2} \otimes \cdots \otimes e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_{\lambda_1}} \).

Let \( f \) be the polynomial defined by the tensor \( \pi^{-1} v_{\lambda} \).

\[
f(y) := \langle \pi^{-1} v_{\lambda}, y \otimes \delta \rangle = \langle v_{\lambda}, \pi(y \otimes \delta) \rangle
\]

\( T_{\lambda} \) is the column-standard tableau.

\[
(\prod_{i} \mu_i ! ) \langle v_{\lambda}, \pi(y \otimes \delta) \rangle \text{ expand} = y \otimes \delta \sum_{\beta: \{1, \ldots, D\delta\} \to \{1, \ldots, m\}, \text{respecting } \pi T_{\lambda}} \prod_{c=1}^{\lambda_1} \det(g_{1..\mu_c, \beta(\pi T_{\lambda}(1,c))}, \ldots, g_{1..\mu_c, \beta(\pi T_{\lambda}(\mu_c,c))}),
\]

where \( \beta \) respects a tableau \( S \) if all numbers \( 1, \ldots, D \) are mapped to the same value, and all numbers \( D + 1, \ldots, 2D \) are mapped to the same value, and so on.

Lemma: \( (\ast) = \langle S_{\lambda}, g(\beta \pi T_{\lambda}) \rangle \)

Therefore the vector space \( \text{HWV}_{\lambda}(C[Gp]) \) is generated by the functions

\[
g \mapsto \sum_{\beta: \{1, \ldots, D\delta\} \to \{1, \ldots, m\}, \text{respecting } S} \langle S_{\lambda}, g\beta(S) \rangle,
\]

where \( S \) runs over all tableaux of shape \( \lambda \) with every number from \( 1, \ldots, D\delta \) exactly once.
Let \( p := x_1^D + \cdots + x_m^D \in \text{Sym}^D \mathbb{C}^m \).

Let \( G := \text{GL}_m \).

Let \( \lambda \vdash \delta D \).

The vector space \( \text{HWV}_\lambda(\mathbb{C}[Gp]) \) is generated by the functions

\[
g \mapsto \sum_{\beta : \{1, \ldots, \delta D\} \to \{1, \ldots, m\}} \langle S_\lambda, g\beta(S) \rangle,
\]

where \( S \) runs over all tableaux of shape \( \lambda \) with every number from \( 1, \ldots, \delta D \) exactly once.

We want:

**Main Theorem of this talk (coordinate ring of the orbit closure of the power sum expressed via tableaux)**

The vector space \( \text{HWV}_\lambda(\mathbb{C}[Gp]) \) is generated by the functions

\[
g \mapsto \langle S_\lambda, gM_{\delta,m}T \rangle,
\]

where \( T \) runs over all semistandard tableaux of shape \( \lambda \) in which each entry \( 1, \ldots, \delta \) appears exactly \( D \) times.

Given a standard tableau \( S \) we define a semistandard tableau \( T \) (or \( T = 0 \)) by replacing the first \( D \) entries \( 1, \ldots, D \) by the number 1, the next \( D \) entries \( D + 1, \ldots, 2D \) by the number 2, and so on.

\[
g \mapsto \sum_{\beta : \{1, \ldots, D\} \to \{1, \ldots, m\}} \langle S_\lambda, g\beta(S) \rangle
\]

\( (1:1 \text{ corresp. between summands}) \)

\[
g \mapsto \sum_{\varphi : \{1, \ldots, \delta\} \to \{1, \ldots, m\}} \langle S_\lambda, g\varphi(T) \rangle
\]

By definition of \( M_{\delta,m} \), this can be rewritten as:

\[
g \mapsto \langle S_\lambda, gM_{\delta,m}T \rangle,
\]

which finishes the proof.
Let $p := x_1^D + \cdots + x_m^D \in \text{Sym}^D \mathbb{C}$. Let $G := \text{GL}_m$. Let $\lambda \vdash \delta D$.

Main Theorem of this talk (coordinate ring of the orbit closure of the power sum expressed via tableaux)

The vector space $\text{HWV}_\lambda(\mathbb{C}[Gp]_\delta)$ is generated by the functions

$$g \mapsto \langle S_\lambda, gM_{\delta,m}T \rangle,$$

where $T$ runs over all semistandard tableaux of shape $\lambda$ in which each entry $1, \ldots, \delta$ appears exactly $D$ times.

In the second talk we will see how to construct such functions, using a similar description for functions in $\mathbb{C}[Gp]$:

Theorem (HWVs in the coordinate ring of the orbit)

The vector space $\text{HWV}_\lambda(\mathbb{C}[Gp]_\delta)$ decomposes into a direct sum of vector spaces $\text{HWV}_\lambda(\mathbb{C}[Gp]_\delta) = \bigoplus_{\varrho \vdash m} \mathcal{W}_\varrho$, and each $\mathcal{W}_\varrho$ is generated by the functions

$$g \mapsto \langle S_\lambda, gP_m \rangle,$$

where $S$ runs over all semistandard tableaux $S$ of shape $\lambda$ and content $\varrho D$.

Thank you for your attention!