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— Workshop —
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Highest weight vectors, power sums, and Young tableaux

(Implementing GCT: On the separation of orbit closures via symmetries)

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- Let $G := \mathsf{GL}_m$, $p = x_1^D + \cdots + x_m^D$.
- We have formulas for $\mathsf{mult}_\lambda(\mathbb{C}[Gp])$.
- We want formulas for $\mathsf{mult}_\lambda(\mathbb{C}[\overline{Gp}])$.

We find a connection between these two via tableau-theoretic interpretations of both multiplicities.

This leads to the first multiplicity obstructions for orbit closure containment, based only on symmetry groups (2nd talk). (with Kandasamy, STOC 2020)

This talk:

A tableau-theoretic interpretation of $\mathsf{mult}_\lambda(\mathbb{C}[\overline{Gp}])$.

- Let $\mathsf{Sym}^D\mathbb{C}^m := \mathbb{C}[x_1,\dots,x_m]_D$.
- Let $G := \mathsf{GL}_m$
- G acts linearly on $\mathsf{Sym}^D\mathbb{C}^m$ via $(gp)(x) := p(g^{-1}x)$.
- Let $X \subseteq \mathsf{Sym}^D\mathbb{C}^m$ be a closed subvariety that is closed under the action of G.
- For example, $X = \overline{Gp}$ an orbit closure.
- Let $\mathsf{Sym}^\delta\mathsf{Sym}^D\mathbb{C}^m := \mathbb{C}[\mathsf{Sym}^D\mathbb{C}^m]_\delta$.

- G acts linearly on $\mathsf{Sym}^\delta\mathsf{Sym}^D\mathbb{C}^m$ via $(gf)(p) := f(g^{-1}p)$, i.e., $\mathsf{Sym}^\delta\mathsf{Sym}^D\mathbb{C}^m$ is a G-representation.
- Let $I(X) = \{f \in \mathbb{C}[\mathsf{Sym}^D\mathbb{C}^m] \mid f(X) = \{0\}\}$ be the vanishing ideal.
- Let $I(X)_\delta = \{f \in \mathsf{Sym}^\delta\mathsf{Sym}^D\mathbb{C}^m \mid f(X) = \{0\}\}$ be the homogeneous degree δ component of the vanishing ideal.
- Since X is closed under the action of G, $I(X)_\delta$ is also closed under the action of G, i.e., $I(X)_\delta$ is a G-representation.
- $\mathbb{C}[X] = \mathbb{C}[\mathsf{Sym}^D\mathbb{C}^m]/I(X)$ is the coordinate ring.
- Alternatively: $\mathbb{C}[X] \simeq \{f|_X \mid f \in \mathsf{Sym}^D\mathbb{C}^m\}$ via restrictions of the domain of definition to X.
- $\mathbb{C}[X]_\delta = \mathsf{Sym}^\delta\mathsf{Sym}^D\mathbb{C}^m/I(X)_\delta$.
- Since $\mathsf{Sym}^\delta\mathsf{Sym}^D\mathbb{C}^m$ is a G-representation and $I(X)_\delta$ is a subrepresentation, it follows that $\mathbb{C}[X]_\delta$ is a G-representation.

- $\mathsf{Sym}^2\mathbb{C}^2 = \langle x_1^2, x_1x_2, x_2^2\rangle$.
- $G = \mathsf{GL}_2$.
- $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)(x_1^2 + x_1x_2) = (x_2^2 + x_1x_2)$.
- $X = \{\ell^2 \mid \exists \alpha,\beta \in \mathbb{C} : \ell = \alpha x_1 + \beta x_2\} \subseteq \mathsf{Sym}^2\mathbb{C}^2$.
- $X = \overline{\mathsf{GL}_2 x_1^2}$.
- $\mathsf{Sym}^2\mathsf{Sym}^2\mathbb{C}^2 = \langle a^2, ab, ac, b^2, bc, c^2\rangle$ $a = (x_1^2)^*, b = (x_1x_2)^*, c = (x_2)^*$
- $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)(a^2 + ac + b^2) = c^2 + ac + b^2$

- $b^2 - 4ac \in I(X)$
- $I(X)_2 = \langle b^2 - 4ac\rangle$

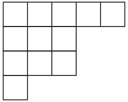
- $g(b^2 - 4ac) = \det(g)^2(b^2 - 4ac)$

- $\dim \mathbb{C}[X]_2 = 6 - 1 = 5$

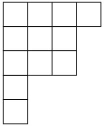
- $\dim \mathbb{C}[X]_2 = \mathsf{Sym}^2\mathsf{Sym}^2\mathbb{C}^2/\langle b^2 - 4ac\rangle$

Young tableaux

- A **partition** is finite nonincreasing list of natural numbers. E.g., $\lambda = (5, 3, 3, 1)$
- The top-left-justified array of boxes with λ_i boxes in row i is called the **Young diagram** of λ.



The transpose λ^t:



- We often identify λ with its Young diagram.
- We write $\lambda \vdash_m d$ if λ is a Young diagram with d boxes and at most m rows.
- For $\lambda \vdash d$, we write $[\lambda]$ for the irreducible $\mathfrak{S}_d$-representation of type λ.
- For $\lambda \vdash_m$, we write $\{\lambda\}$ for the irreducible GL_m-representation of type λ.
- A **Young tableau** or just **tableau** $T : \lambda \to \mathbb{N}$ is an assignment of numbers to the boxes of λ.

1	1	1	2	2
2	3	3		
3	4	4		
4				

- A Young tableau is called **semistandard** if each column is increasing from top to bottom and every row is nondecreasing from left to right.

The **superstandard tableau** $\mathbb{S}_\lambda$ of shape λ has only entries i in row i:

1	1	1	1	1
2	2	2		
3	3	3		
4				

The **column-standard** tableau T_λ of shape λ:

1	5	8	11	12
2	6	9		
3	7	10		
4				

Multiplicities and highest weight polynomials

- Every GL_m-representation $\mathscr{V}$ decomposes (not necessarily uniquely) into a direct sum of irreducible subrepresentations:

$$\mathscr{V} = \bigoplus_{\lambda \vdash m} \{\lambda\}^{\oplus c_\lambda}$$

  For example:

$$\mathsf{Sym}^\delta \mathsf{Sym}^D \mathbb{C}^m = \bigoplus_{\lambda \vdash m} \{\lambda\}^{\oplus a_\lambda(\delta, D)}$$

- These nonnegative integers c_λ (and hence also $a_\lambda(\delta, D)$) are independent of the decomposition.
  They are called **representation theoretic multiplicities**. $\mathsf{mult}_\lambda(\mathscr{V}) := c_\lambda$.
  $a_\lambda(\delta, D)$ is called a **plethysm coefficient**.

Every irreducible GL_m-representation contains exactly one (up to scale) **highest weight vector** (HWV):

- $\mathsf{diag}(\alpha_1, \dots, \alpha_m) f = \alpha_1^{\lambda_1} \cdots \alpha_m^{\lambda_m} f$
- $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} f = f$

The exponent vector λ is called the **weight** of f.
Two irreducible GL_m-representations are **isomorphic** iff the weights of their HWVs coincide.

Example: $g(b^2 - 4ac) = \mathsf{det}(g)^2 (b^2 - 4ac)$, hence $b^2 - 4ac$ is a HWV of weight $(2, 2)$.

- $\mathsf{mult}_\lambda(\mathscr{V}) = \dim \mathsf{HWV}_\lambda(\mathscr{V})$
- $\mathsf{mult}_\lambda(\mathbb{C}[X]_\delta) = \dim \Big((\mathsf{HWV}_\lambda(\mathsf{Sym}^\delta \mathsf{Sym}^D \mathbb{C}^m))|_X \Big)$.

Some papers with coauthors where HWVs are explicitly constructed from Young tableaux:

1. *16,051 formulas for Ottaviani's invariant of cubic threefolds*, with Abdesselam and Royle 2014: $\lambda = (9, 9, 9, 9, 9)$, $D = 3$, $\delta = 15$.
2. *Symmetrizing Tableaux and the 5th case of the Foulkes Conjecture*, with Cheung and Mkrtchyan 2015: $(\delta, D) = (5, 5), (5, 6), (6, 6)$.
3. *Fundamental invariants of orbit closures*, with Bürigsser 2015: rectangular λ.
4. *On GCT: Multiplicity obstructions are stronger than occurrence obstructions*, with Dörfler and Panova 2019: $D = 6, 7$.
5. *On the complexity of evaluating highest weight vectors*, with Bläser and Dörfler 2020: HWV evaluation is NP-hard. Efficient algorithms for evaluation for low tree-width.
6. *Equations for GL invariant families of polynomials* with Breiding, Michałek, Hodges 2021: $D = 3$, $\delta = 11$, $\lambda = (15, 6, 6, 6)$. Software and database. Uses Hodges' fast tableau straightening code, arXiv:1710.05214
7. *Implementing GCT: On the separation of orbit closures via symmetries*, with Kandasamy 2019: HWV construction is implicit from the existence of certain invariants.

Many other authors, because this is a very classical approach.

In this talk we study the power sum:

- $p := x_1^D + \cdots + x_m^D \quad \in \quad \mathsf{Sym}^D\mathbb{C}^m.$
- $G := \mathsf{GL}_m.$

We want to understand $\mathsf{HWV}_\lambda(\mathbb{C}[\overline{Gp}]_\delta)$ in terms of Young tableaux.

$T(r,c)$: entry of T in row r and column c. Let $\{e_i\} \subset \mathbb{C}^m$ standard basis. Let $\mu = \lambda^t$.

tableau $T : \lambda \to \{1,\dots,m\}$

$\updownarrow$ 1:1 correspondence

basis vector $e_{T(1,1)} \otimes e_{T(2,1)} \otimes \cdots \otimes e_{T(\mu_1,1)} \otimes e_{T(1,2)} \otimes e_{T(2,2)} \otimes \cdots \otimes e_{T(\mu_2,2)} \otimes \cdots\cdots \otimes e_{T(\mu_{\lambda_1},\lambda_1)} \in \bigotimes^{|\lambda|}\mathbb{C}^m$

This gives a vector space isomorphism: {linear combinations of tableaux of shape λ} $\xrightarrow{\sim} \bigotimes^{|\lambda|}\mathbb{C}^m$.

The G-action, which is given by $g(v_1 \otimes \cdots \otimes v_{|\lambda|}) = g(v_1) \otimes \cdots \otimes g(v_{|\lambda|})$, is inherited:

Example: $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&1\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&2\\\hline 2&2\\\hline\end{array}.$

The vector space of shuffle relations (Grassmann-Plücker relations)

The subrepresentation $K(\lambda)$ is spanned by

- $T + T'$, where T' is a tableau that arises from T by switching two numbers within one column.
- $T - \Sigma S$, where for two fixed columns j, j' and a fixed number of entries k the sum is over all tableaux S that arise from T by exchanging the top k entries in column j with any k entries in column j', preserving the internal vertical order.

Theorem

$\{\lambda\} \simeq (\bigotimes^{|\lambda|}\mathbb{C}^m)/K(\lambda)$. Basis: the semistd tableaux of shape λ with entries from $\{1,\dots,m\}$. HWV: $\mathbb{S}_\lambda$

$\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} = -\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$; $\begin{array}{|c|c|}\hline 4&1\\\hline 2\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 4&2\\\hline 1\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}$; $\begin{array}{|c|c|}\hline 4&1\\\hline 2&2\\\hline\end{array} = \begin{array}{|c|c|}\hline 1&4\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 4&2\\\hline 1&2\\\hline\end{array} = \begin{array}{|c|c|}\hline 1&4\\\hline 2&2\\\hline\end{array}$; $\begin{pmatrix}1 & \alpha & \beta\\ 0 & 1 & \gamma\\ 0 & 0 & 1\end{pmatrix} \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2&2\\\cline{1-3} 3\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2&2\\\cline{1-3} 3\\\cline{1-1}\end{array}$

If we insist that every $1,\dots,m$ appears exactly once, we obtain the Specht modules $[\lambda]$, i.e., the irreps of $\mathfrak{S}_m$.

Averaging

- Let $T : \lambda \to \{1, \ldots, \delta\}$ be a tableau.
- Let $\varphi : \{1, \ldots, \delta\} \to \{1, \ldots, m\}$ be a map.
- Define $\varphi T := \varphi(T)$ naively.

Example: $\begin{bmatrix} \varphi(1) = 2 \\ \varphi(2) = 2 \\ \varphi(3) = 1 \end{bmatrix}$ implies $\varphi\left(\begin{array}{ccccc} 1&1&1&2&3 \\ 2&2&3&& \\ 3&&&& \end{array}\right) = \underbrace{\begin{array}{ccccc} 2&2&2&2&1 \\ 2&2&1&& \\ 1&&&& \end{array}}_{=0 \text{ in } \{(5,3,1)\}}$.

The "orbit average": $\quad M_{\delta,m}T := \sum_{\varphi:\{1,\ldots,\delta\}\to\{1,\ldots,m\}} \varphi T \quad \in \{\lambda\}^{\mathfrak{S}_m}$

Example: $M_{3,2} \begin{array}{ccccc} 1&1&1&2&3 \\ 2&2&3&& \\ 3&&&& \end{array} = \begin{array}{ccccc} 1&1&1&2&1 \\ 2&2&1&& \\ 1&&&& \end{array} + \begin{array}{ccccc} 1&1&1&2&2 \\ 2&2&2&& \\ 2&&&& \end{array} + \begin{array}{ccccc} 2&2&2&2&1 \\ 2&2&1&& \\ 1&&&& \end{array} + \begin{array}{ccccc} 2&2&2&2&2 \\ 2&2&2&& \\ 2&&&& \end{array} + \begin{array}{ccccc} 1&1&1&1&1 \\ 1&1&1&& \\ 1&&&& \end{array} + \begin{array}{ccccc} 1&1&1&1&2 \\ 1&1&2&& \\ 2&&&& \end{array} + \begin{array}{ccccc} 2&2&2&1&1 \\ 1&1&1&& \\ 1&&&& \end{array} + \begin{array}{ccccc} 2&2&2&1&2 \\ 1&1&2&& \\ 2&&&& \end{array} = 0$

Main Theorem of this talk (coordinate ring of the orbit closure of the power sum expressed via tableaux)

Let $p := x_1^D + \cdots + x_m^D \in \mathsf{Sym}^D\mathbb{C}^m$. Let $G := \mathsf{GL}_m$.
Let $\lambda \vdash \delta D$. The vector space $\mathsf{HWV}_\lambda(\mathbb{C}[\overline{Gp}]_\delta)$ is generated by the functions

$$g \mapsto \langle \mathbb{S}_\lambda, gM_{\delta,m}T \rangle , \qquad \langle S, T \rangle = \delta_{S,T}$$

where T runs over all semistandard tableaux of shape λ in which each entry $1, \ldots, \delta$ appears exactly D times.

$M_{\delta,m}T := \sum_{\varphi:\{1,\dots,\delta\}\to\{1,\dots,m\}} \varphi(T)$
Let $p := x_1^D + \cdots + x_m^D \in \mathsf{Sym}^D\mathbb{C}^m$. Let $G := \mathsf{GL}_m$. Let $\lambda \vdash \delta D$.

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$$g \mapsto \langle \mathbb{S}_\lambda, gM_{\delta,m}T\rangle, \qquad \langle S,T\rangle = \delta_{S,T}$$

where T runs over all semistandard tableaux of shape λ in which each entry $1,\dots,\delta$ appears exactly D times.

Well-definedness:
$\mathsf{stab}(p) \simeq C_D \wr \mathfrak{S}_m$
For $q \in Gp$ we have $q = gp$, but g is unique only up to right-mult. with $\mathsf{stab}(p)$: $gp = ghp$ with $h \in \mathsf{stab}(p)$.

Example ($D=3$, $\delta=4$, $m=2$)

The cyclic group part: If $\omega^3 = 1$, then $\mathrm{diag}(\omega,1)\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&1&1&1\\ \hline 2&2&1\\ \cline{1-3}\end{array} = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&1&1&1\\ \hline 2&2&1\\ \cline{1-3}\end{array}$

The symmetric group part: If $h \in \mathfrak{S}_m$, then $hM_{\delta,m}T$ and $M_{\delta,m}T$ are the same sum, just permuted order of summands.

Main Theorem of this talk (coordinate ring of the orbit closure of the power sum expressed via tableaux)

Let $p := x_1^D + \cdots + x_m^D \in \mathsf{Sym}^D\mathbb{C}^m$. Let $G := \mathsf{GL}_m$.
Let $\lambda \vdash \delta D$. The vector space $\mathsf{HWV}_\lambda(\mathbb{C}[\overline{Gp}]_\delta)$ is generated by the functions

$$g \mapsto \langle \mathbb{S}_\lambda, gM_{\delta,m}T\rangle, \qquad \langle S,T\rangle = \delta_{S,T}$$

where T runs over all semistandard tableaux of shape λ in which each entry $1,\ldots,\delta$ appears exactly D times.

Discriminant $T = \frac{1}{4}\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}$

Discriminant at x_1^2 is zero: $M_{2,1}T = \frac{1}{4}\begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array} = 0$. It follows that it vanishes on Waring rank 1.

Discriminant at $x_1^2+x_2^2$ is nonzero: $M_{2,2}T = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&2\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array} = 2\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}$, $\quad \left\langle \frac{1}{4}\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}, 2\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\right\rangle = \frac{1}{2} \neq 0$

Aronhold invariant: The HWV to

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2&3&3\\\hline 3&4&4&4\\\hline\end{array}$$

vanishes on Waring rank 3. But one calculates that it does not vanish on $x_1^3+x_2^3+x_3^3+(x_1+x_2+x_3)^3 \in \overline{\mathsf{GL}_4(x_1^3+\cdots+x_4^3)}$.

Let $p := x_1^D + \cdots + x_m^D \in \mathsf{Sym}^D \mathbb{C}^m$. Let $G := \mathsf{GL}_m$. Let $\lambda \vdash \delta D$.

Main Theorem of this talk (coordinate ring of the orbit closure of the power sum expressed via tableaux)

The vector space $\mathsf{HWV}_\lambda(\mathbb{C}[\overline{Gp}]_\delta)$ is generated by the functions

$$g \mapsto \langle \mathbb{S}_\lambda, gM_{\delta,m}T \rangle \,,$$

where T runs over all semistandard tableaux of shape λ in which each entry $1, \ldots, \delta$ appears exactly D times.

In the second talk we will see how to construct such functions, using a similar description for functions in $\mathbb{C}[Gp]$:

Theorem (HWVs in the coordinate ring of the orbit)

The vector space $\mathsf{HWV}_\lambda(\mathbb{C}[Gp]_d)$ decomposes into a direct sum of vector spaces $\mathsf{HWV}_\lambda(\mathbb{C}[Gp]_d) = \bigoplus_{\varrho \vdash_m d} \mathscr{W}_\varrho$, and each $\mathscr{W}_\varrho$ is generated by the functions

$$g \mapsto \langle \mathbb{S}_\lambda, gP_m S \rangle \,,$$

where S runs over all semistandard tableaux S of shape λ and content ϱD.

$$\begin{array}{ccccc} \bigotimes^{\delta D}\mathbb{C}^m & \twoheadrightarrow & \mathsf{Sym}^\delta \mathsf{Sym}^D \mathbb{C}^m & \twoheadrightarrow & \mathbb{C}[\overline{Gp}]_\delta \\ \cup\!\!| & & \cup\!\!| & & \cup\!\!| \\ \mathsf{HWV}_\lambda(\bigotimes^{\delta D}\mathbb{C}^m) & \twoheadrightarrow & \mathsf{HWV}_\lambda(\mathsf{Sym}^\delta \mathsf{Sym}^D \mathbb{C}^m) & \twoheadrightarrow & \underbrace{\mathsf{HWV}_\lambda(\mathbb{C}[\overline{Gp}]_\delta)}_{\text{our goal}} \end{array}$$

Theorem (Schur-Weyl duality)

$$\bigotimes^{\delta D}\mathbb{C}^m = \bigoplus_{\lambda \vdash_m \delta D} \{\lambda\} \otimes [\lambda], \qquad \mathsf{HWV}_\nu(\bigotimes^{\delta D}\mathbb{C}^m) \simeq [\nu]$$

This implies: If we find $v_\lambda \in \mathsf{HWV}_\lambda(\bigotimes^{\delta D}\mathbb{C}^m)$, then

$$\mathsf{HWV}_\lambda(\bigotimes^{\delta D}\mathbb{C}^m) = \mathsf{linspan}(\{\pi v_\lambda \mid \pi \in \mathfrak{S}_{\delta D}\}).$$

An example of such a $v_\lambda \in \mathsf{HWV}_\lambda(\bigotimes^{\delta D}\mathbb{C}^m)$:

$$v_\lambda := e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_1} \otimes e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_2} \otimes \cdots\cdots \otimes e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_{\lambda_1}},$$

where $\mu = \lambda^t$.

Example: $e_1 \wedge e_2 \otimes e_1 \wedge e_2 \;=\; \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1) \otimes \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1)$

$$= \frac{1}{4}(e_1 \otimes e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_2 \otimes e_1) \in \mathsf{HWV}_{(2,2)}(\bigotimes^4 \mathbb{C}^2)$$

$$\downarrow (2\,3)$$

$$\frac{1}{4}(e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_2 \otimes e_1 \otimes e_1) \in \mathsf{HWV}_{(2,2)}(\bigotimes^4 \mathbb{C}^2)$$

Defining polynomials via tensor contraction

Let $v \in \bigotimes^{\delta D}\mathbb{C}^m$.

$$\forall y \in \mathsf{Sym}^D\mathbb{C}^m:\ f(y) := \langle v, y^{\otimes\delta}\rangle \text{ defines a polynomial } f \in \mathsf{Sym}^\delta\mathsf{Sym}^D\mathbb{C}^m.$$

- Example (evaluation at $x_1^2 + x_2^2$, which as a tensor is $e_1 \otimes e_1 + e_2 \otimes e_2$):

$$\langle \tfrac{1}{4}(e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_2 \otimes e_1 \otimes e_1), (e_1 \otimes e_1 + e_2 \otimes e_2)^{\otimes 2}\rangle$$
$$= \tfrac{1}{4}\langle 1122 - 2112 - 1221 + 2211,\ 1111 + 1122 + 2211 + 2222\rangle = \tfrac{1}{4}\cdot 2 = \tfrac{1}{2}$$

Evaluation at x_1x_2, which as a tensor is $\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)$:

$$\langle \tfrac{1}{4}(e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_2 \otimes e_1 \otimes e_1), (\tfrac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1))^{\otimes 2}\rangle$$
$$= \tfrac{1}{16}\langle 1122 - 2112 - 1221 + 2211,\ 1212 + 1221 + 2112 + 2121\rangle = \tfrac{1}{16}\cdot(-2) = -\tfrac{1}{8}$$

Remark that we recognize this: $-\frac{1}{8}b^2 + \frac{1}{2}ac = -\frac{1}{4}(b^2 - 4ac)$, the discriminant.

- Alternative way to calculate via keeping the determinants:

$$\langle (2\,3)(1 \wedge 2 \otimes 1 \wedge 2)\ ,\ 1111 + 1122 + 2211 + 2222\rangle = \langle 1 \wedge 2 \otimes 1 \wedge 2\ ,\ 1111 + 1212 + 2121 + 2222\rangle$$
$$= \tfrac{1}{4}\left(\det\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix} + \det\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \det\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \det\begin{pmatrix}0&0\\1&1\end{pmatrix}\begin{pmatrix}0&0\\1&1\end{pmatrix}\right) = \tfrac{1}{2}$$

- Alternative way to calculate via contraction with superstandard tableau (this talk's main theorem):

$$T = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array},\ \mathbb{S}_{2,2} = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array},\quad M_{2,2}T = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&2\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array} = 2\cdot\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}. \qquad \frac{1}{\prod_i \mu_i!}\langle \mathbb{S}_{2,2}, M_{2,2}T\rangle = \tfrac{1}{4}\cdot 2 = \tfrac{1}{2}.$$

Evaluation at the power sum

Let $p := x_1^D + \cdots + x_m^D \in \mathsf{Sym}^D\mathbb{C}^m$. Let $gp = \ell_1^D + \cdots + \ell_m^D$, i.e., $\ell_i = g_{1..m,i}$. Let $y := gp$.
We calculate

$$y^{\otimes\delta} = \sum_{\varphi:\{1,\dots,\delta\}\to\{1,\dots,m\}} \ell_{\varphi(1)}^D \otimes \cdots \otimes \ell_{\varphi(\delta)}^D.$$

$$\text{Recall} \quad v_\lambda := e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_1} \otimes e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_2} \otimes \cdots\cdots \otimes e_1 \wedge e_2 \wedge \cdots \wedge e_{\mu_{\lambda_1}}.$$

Let f be the polynomial defined by the tensor $\pi^{-1}v_\lambda$.

$$f(y) := \langle \pi^{-1}v_\lambda, y^{\otimes\delta}\rangle = \langle v_\lambda, \pi(y^{\otimes\delta})\rangle$$

T_λ is the column-standard tableau.

$$\left(\prod_i \mu_i!\right)\langle v_\lambda, \pi(y^{\otimes\delta})\rangle \overset{\text{expand } y^{\otimes\delta}}{=} \sum_{\substack{\beta:\{1,\dots,\delta D\}\to\{1,\dots,m\} \\ \text{respecting } \pi T_\lambda}} \underbrace{\prod_{c=1}^{\lambda_1} \det(g_{1..\mu_c,\beta(\pi T_\lambda(1,c))}, \dots, g_{1..\mu_c,\beta(\pi T_\lambda(\mu_c,c))})}_{=:(*)},$$

where β **respects** a tableau S if all numbers $1,\dots,D$ are mapped to the same value, and all numbers $D+1,\dots,2D$ are mapped to the same value, and so on.

Lemma: $(*) = \langle \mathbb{S}_\lambda, g(\beta\pi T_\lambda)\rangle$

Therefore the vector space $\mathsf{HWV}_\lambda(\mathbb{C}[\overline{Gp}])$ is generated by the functions

$$g \mapsto \sum_{\substack{\beta:\{1,\dots,D\delta\}\to\{1,\dots,m\} \\ \text{respecting } S}} \langle \mathbb{S}_\lambda, g\beta(S)\rangle,$$

where S runs over all tableaux of shape λ with every number from $1,\dots,D\delta$ exactly once.

Let $p := x_1^D + \cdots + x_m^D \in \mathsf{Sym}^D\mathbb{C}^m$. Let $G := \mathsf{GL}_m$. Let $\lambda \vdash \delta D$.

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where S runs over all tableaux of shape λ with every number from $1,\ldots,D\delta$ exactly once.

We want:

Main Theorem of this talk (coordinate ring of the orbit closure of the power sum expressed via tableaux)

The vector space $\mathsf{HWV}_\lambda(\mathbb{C}[\overline{Gp}]_\delta)$ is generated by the functions

$$g \mapsto \left\langle \mathbb{S}_\lambda, gM_{\delta,m}T\right\rangle,$$

where T runs over all semistandard tableaux of shape λ in which each entry $1,\ldots,\delta$ appears exactly D times.

Given a standard tableau S we define a semistandard tableau T (or $T = 0$) by replacing the first D entries $1,\ldots,D$ by the number 1, the next D entries $D+1,\ldots,2D$ by the number 2, and so on.

$$g \mapsto \sum_{\substack{\beta:\{1,\ldots,D\delta\}\to\{1,\ldots,m\} \\ \text{respecting } S}} \langle \mathbb{S}_\lambda, g\beta(S)\rangle \quad \text{can be rewritten as (1:1 corresp. between summands)} \quad g \mapsto \sum_{\varphi:\{1,\ldots,\delta\}\to\{1,\ldots,m\}} \langle \mathbb{S}_\lambda, g\varphi(T)\rangle$$

By definition of $M_{\delta,m}$, this can be rewritten as:

$$g \mapsto \langle \mathbb{S}_\lambda, gM_{\delta,m}T\rangle,$$

which finishes the proof. □

Let $p := x_1^D + \cdots + x_m^D \in \mathsf{Sym}^D \mathbb{C}^m$. Let $G := \mathsf{GL}_m$. Let $\lambda \vdash \delta D$.

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where T runs over all semistandard tableaux of shape λ in which each entry $1, \ldots, \delta$ appears exactly D times.

In the second talk we will see how to construct such functions, using a similar description for functions in $\mathbb{C}[Gp]$:

Theorem (HWVs in the coordinate ring of the orbit)

The vector space $\mathsf{HWV}_\lambda(\mathbb{C}[Gp]_d)$ decomposes into a direct sum of vector spaces $\mathsf{HWV}_\lambda(\mathbb{C}[Gp]_d) = \bigoplus_{\varrho \vdash_m d} \mathscr{W}_\varrho$, and each $\mathscr{W}_\varrho$ is generated by the functions

$$g \mapsto \langle \mathbb{S}_\lambda, gP_m S \rangle ,$$

where S runs over all semistandard tableaux S of shape λ and content ϱD.

Thank you for your attention!