ALGEBRAIC COMPLEXITY: AN INTRODUCTION

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Polynomials

$$P_n(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n] \quad , \quad n \geq 1$$

$$P_n(\bar{x}) = \sum_{\bar{e} = (e_1, \ldots, e_n)} \alpha_{\bar{e}} \, x^{\bar{e}}$$

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Defines a computational problem

Input: $a \in \mathbb{F}^n$

Output: $P_n(a) \in \mathbb{F}$

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Captures many interesting computational problems.

Examples

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→ $E_n^d(x_1, \ldots, x_n) = \sum_{S \subseteq [n],\, |S|=d} \prod_{i \in S} x_i$

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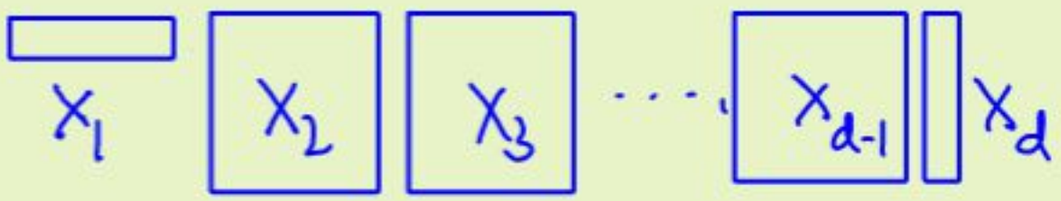
→ $IMM_n^d(X_1, \ldots, X_d) =$ 

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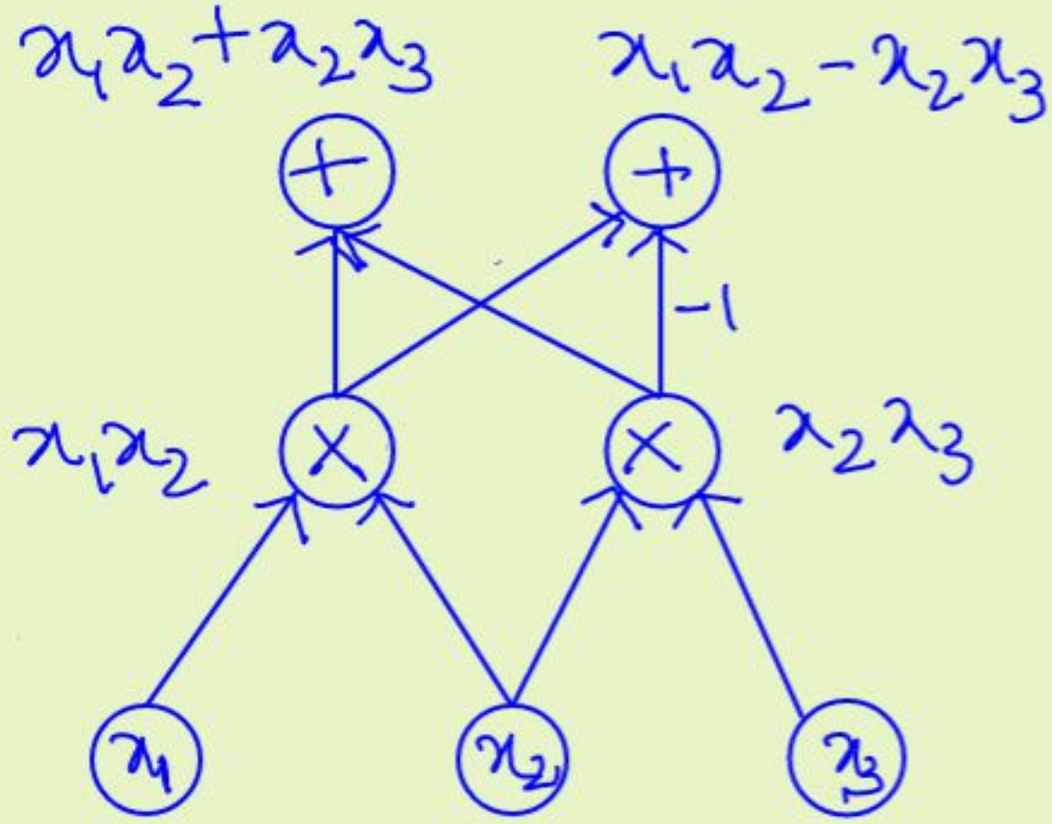
→ True for many (most?) naturally occurring problems.

Algebraic circuits



→ One or more output gates.

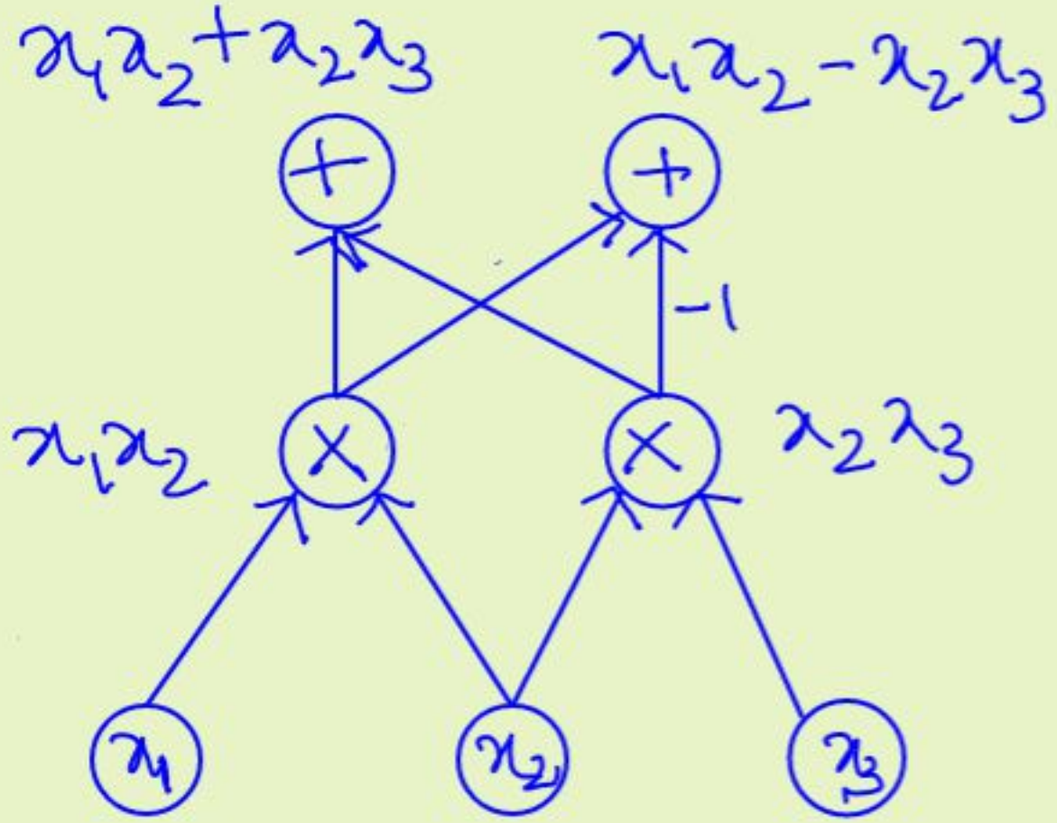
→ labelled DAG

→ Leaves ↔ variables, constants

→ × - gates for product

→ + - gates for sums (linear combinations)

Algebraic circuits



Size = # of edges/wires

Depth = length of longest input-output path.

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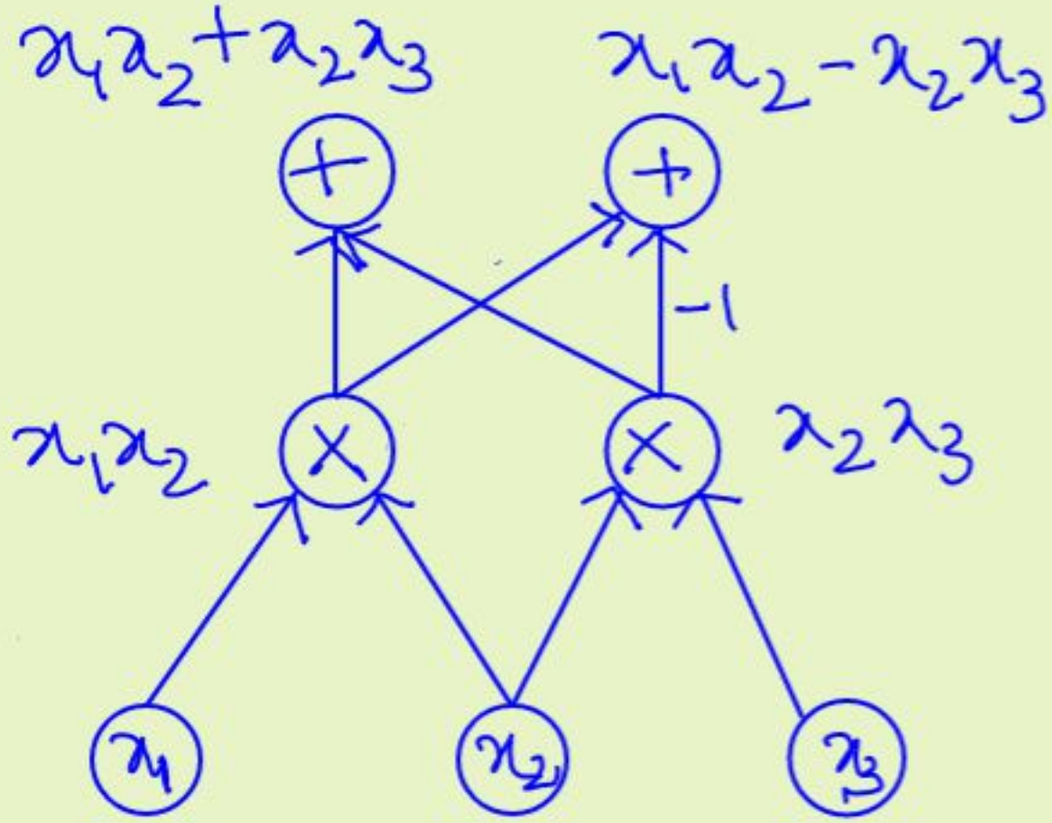


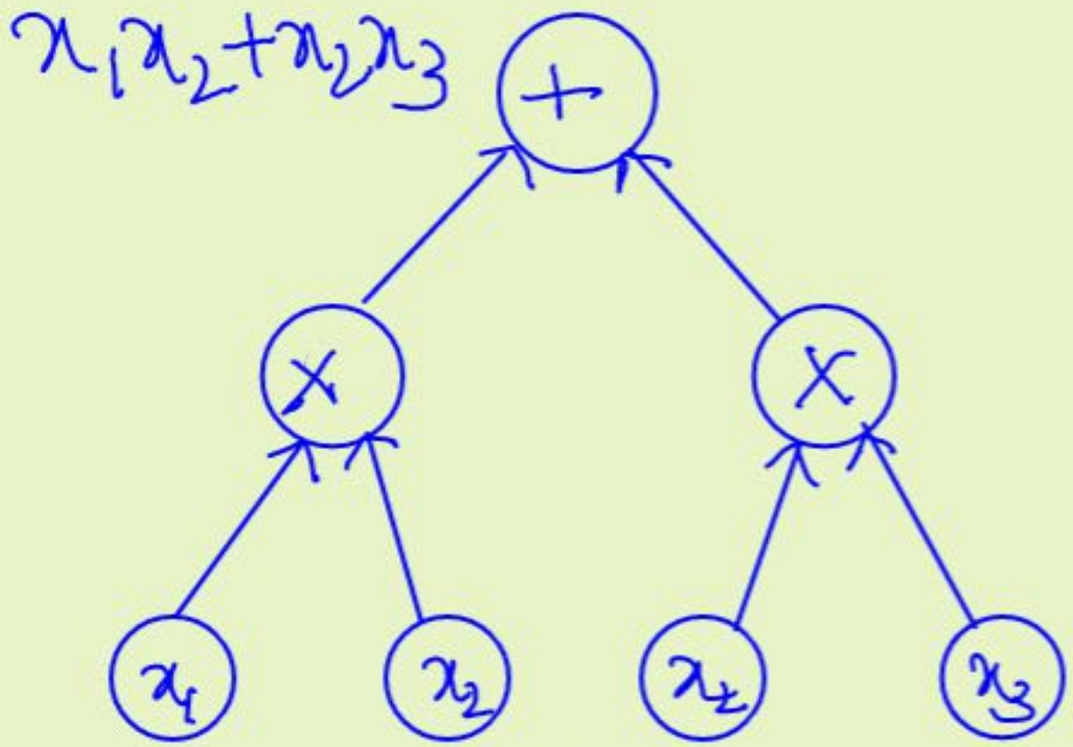
Size = 8

Depth = 2

Size = # of edges/wires

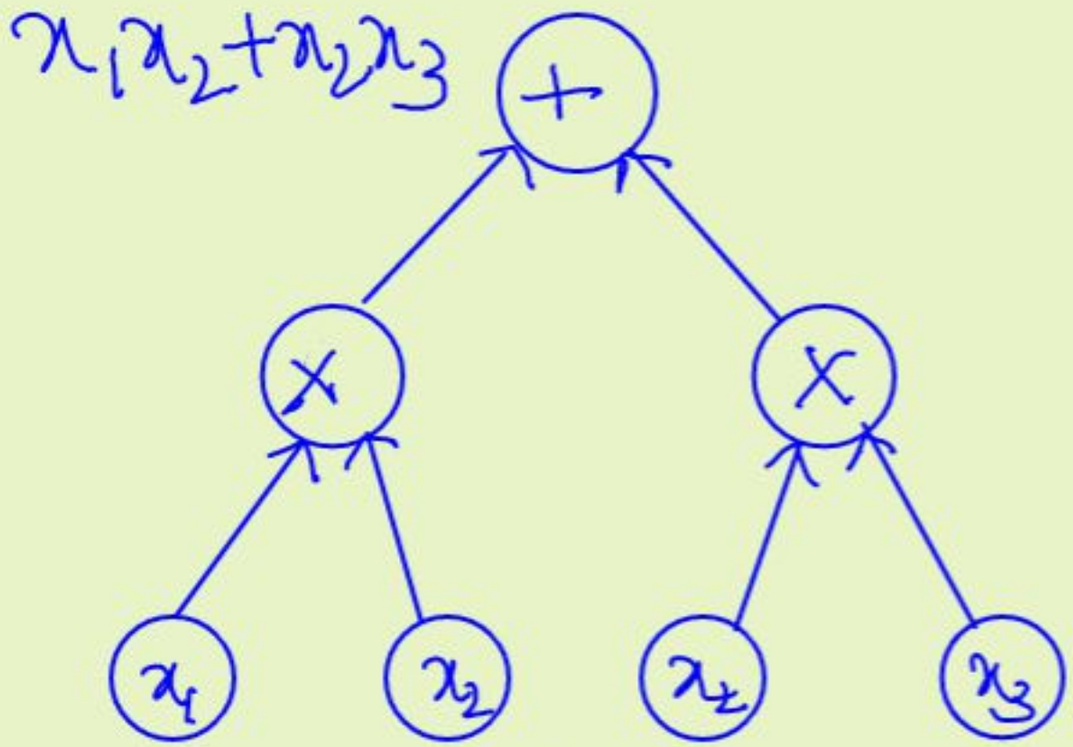
Depth = length of longest input-output path.

Algebraic formulas



→ Underlying graph is a tree

Algebraic formulas



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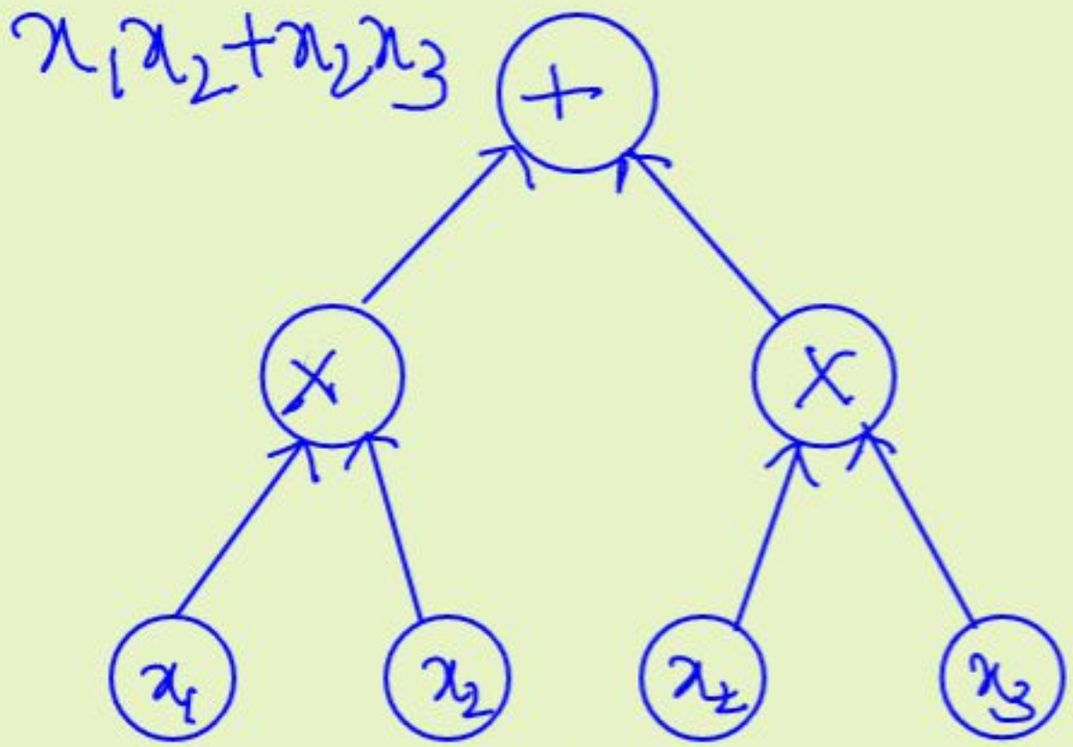

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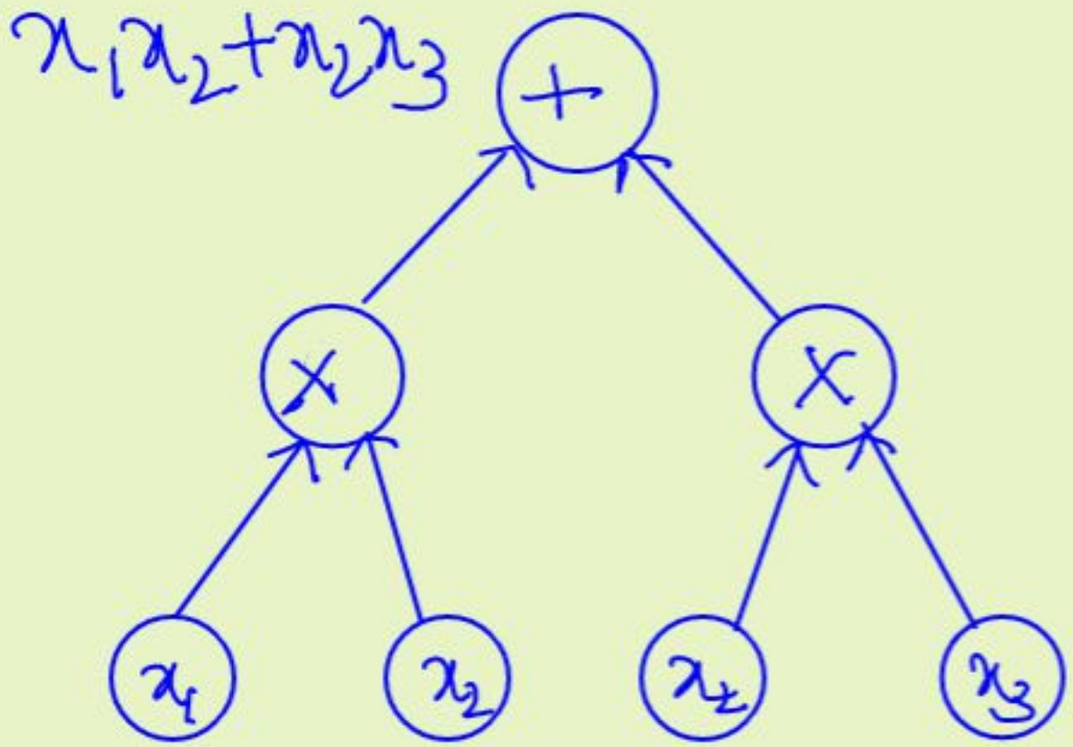
Algebraic formulas



→ Underlying graph is a tree

→ Formulas ⟷ algebraic expressions

→ Size of formula ≈ Size of expression

→ Any $P \in \mathbb{F}[x_1, \ldots, x_n]$ has a "trivial" depth-2 formula of size $\binom{n+d}{d}$

Reductions

→ (P_n) & (Q_m)

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$$P_n(x_1, \ldots, x_n) = Q_{m(n)}(l_1(\bar{x}), \ldots, l_{m(n)}(\bar{x}))$$

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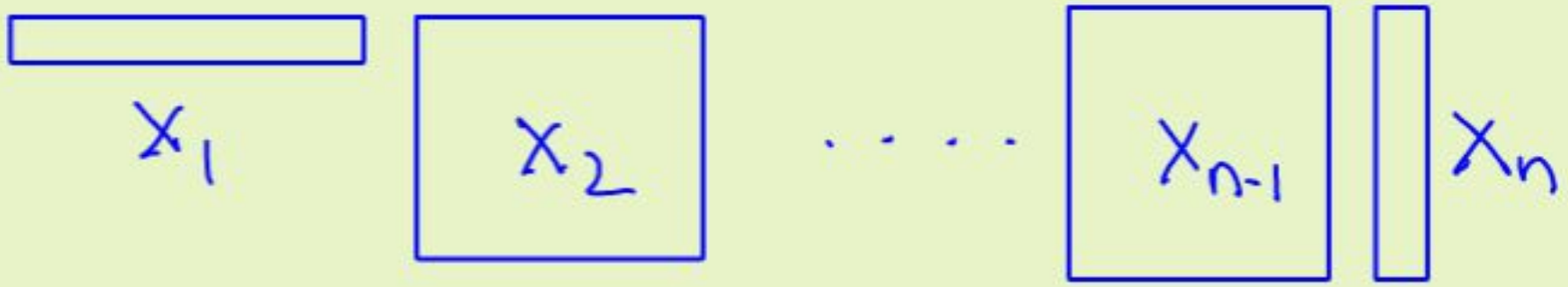
→ "Reducing" one computational problem to another.

Reductions to IMM

$IMM_{n,n} =$



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$$= \sum_{i_1, i_2 \cdots, i_{n-1}} X^{(1)}_{i_1} X^{(2)}_{i_1,i_2} X^{(3)}_{i_2,i_3} \cdots X^{(n-1)}_{i_{n-1},i_n} \cdot X^{(n)}_{i_n}$$

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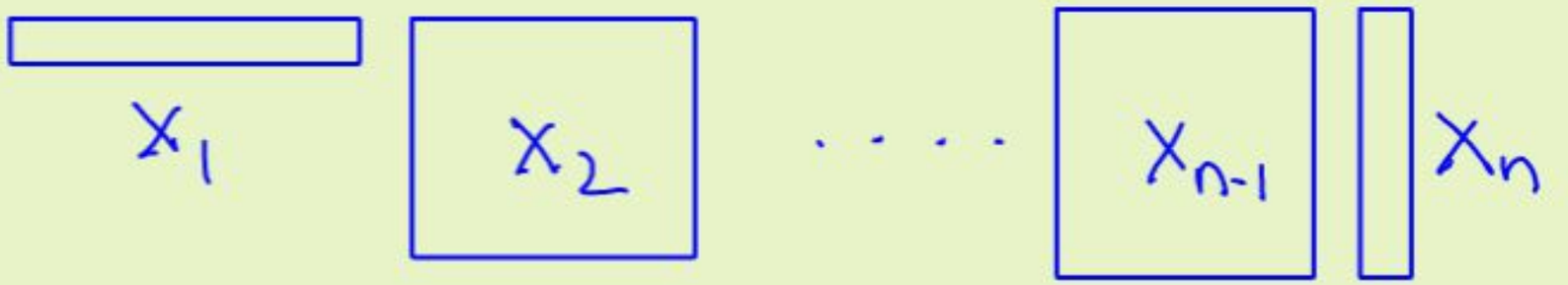

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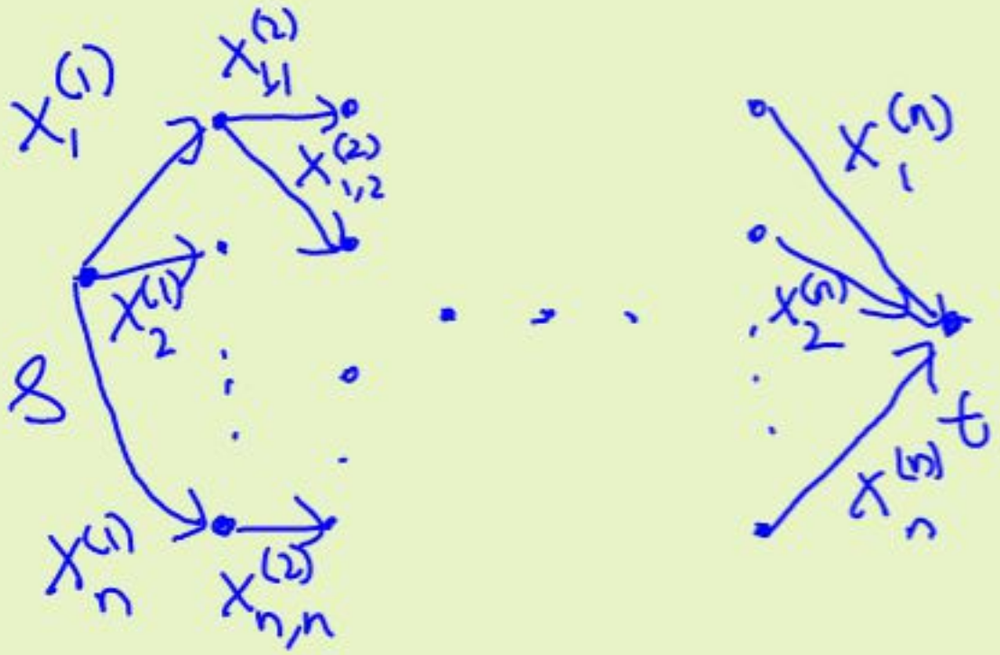

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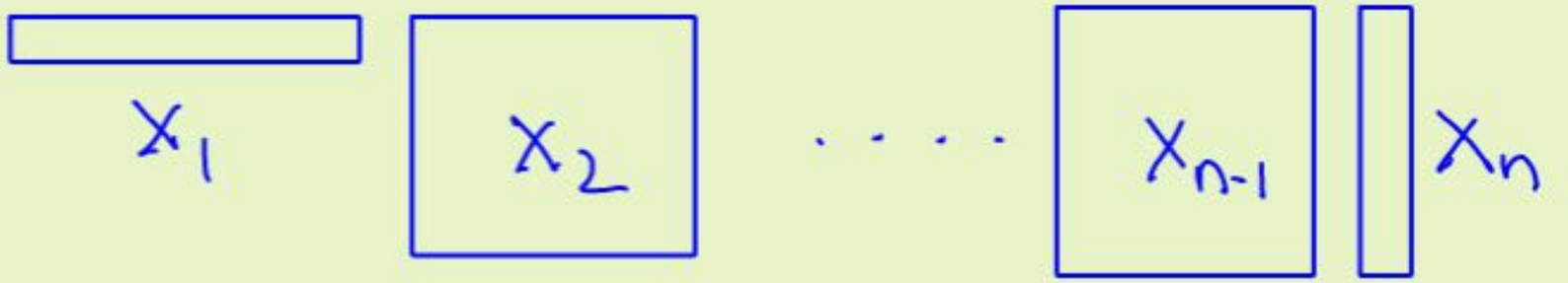

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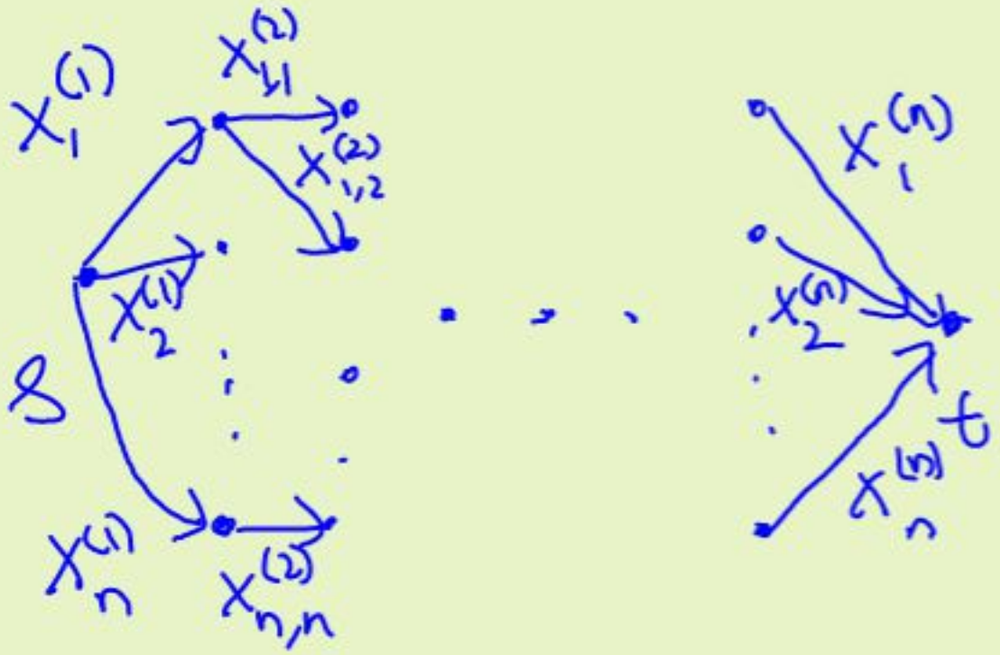


$IMM_{n,n} =$

$$\sum_{s\text{-}t \text{ paths } p} \prod_{\text{edges } e \text{ of } p} \text{label}(e)$$

Algebraic Branching Programs (ABPs)

→ Layered DAG

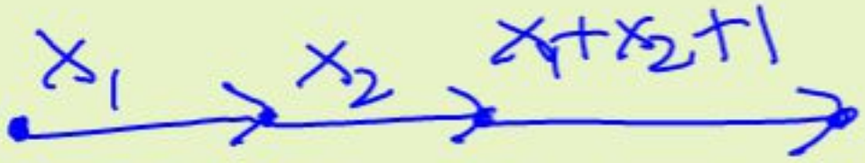
source s &

sink t

→ Edges labelled by linear polys

$$P_G = \sum_{\substack{p : s \to t \\ \text{path}}} \prod_{e \text{ on } p} l_e$$

Examples of ABPs

$G_1 =$ 

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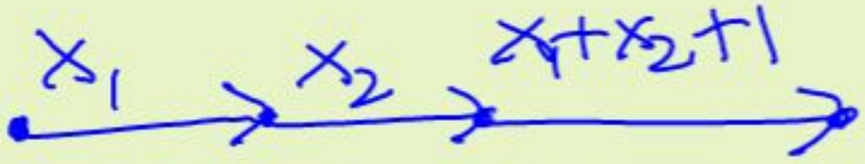
$G_1 =$ 

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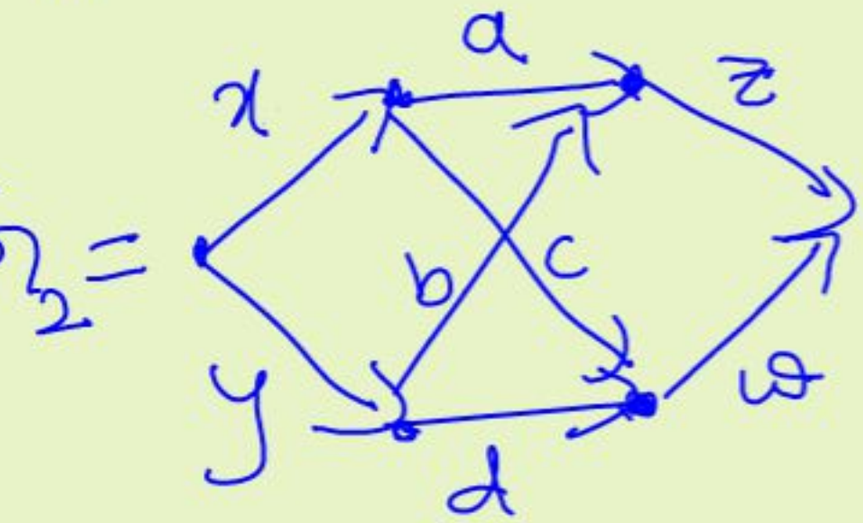
$G_2 =$ 

$P_{G_2} = xaz + xcw + ybz + ydw$

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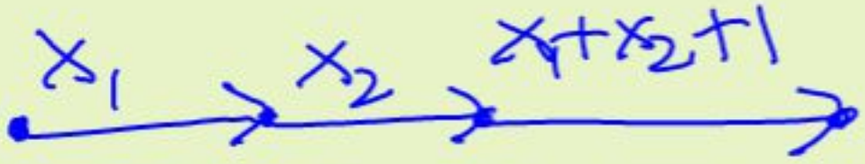
$G_1 =$ 

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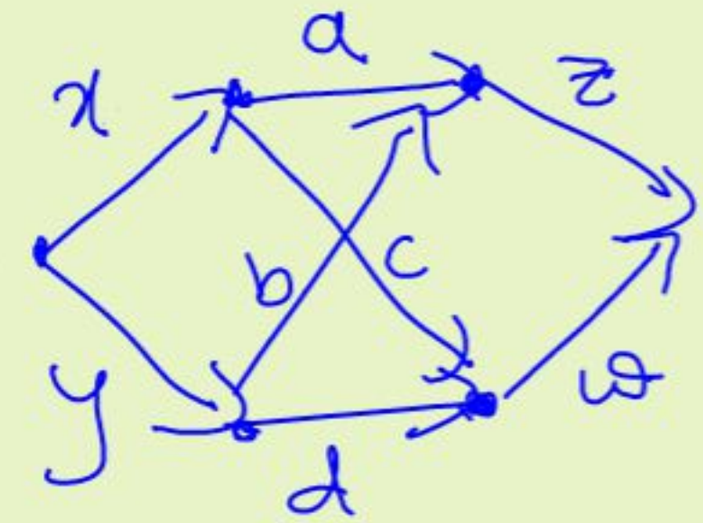
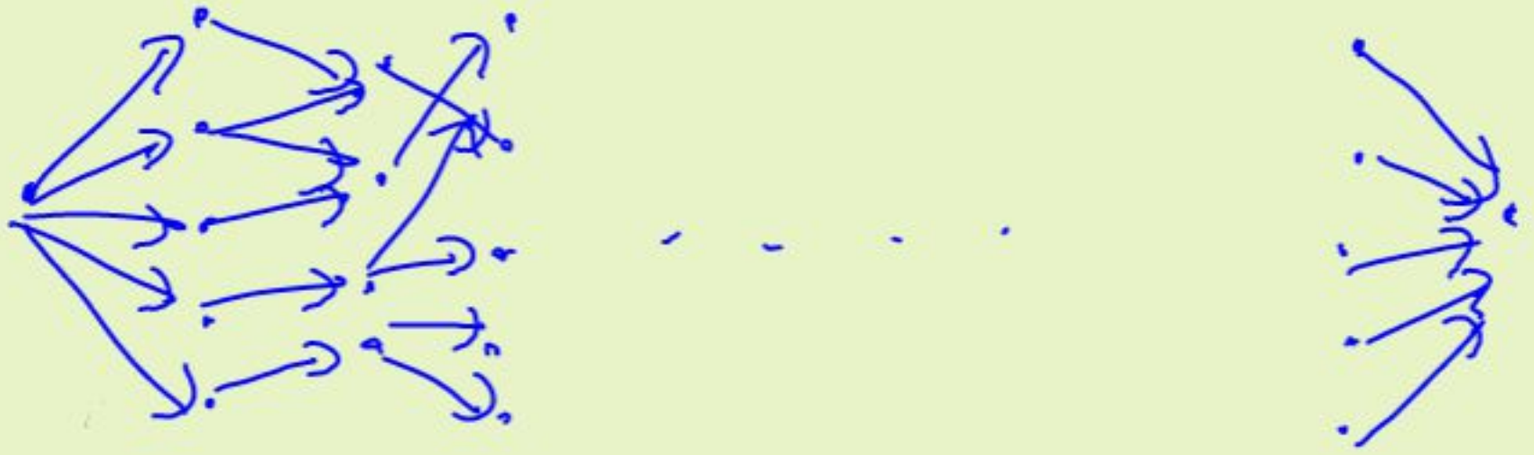
$G_2 =$ 

$P_{G_2} = \lambda a z + x c w + y b z + y d w$

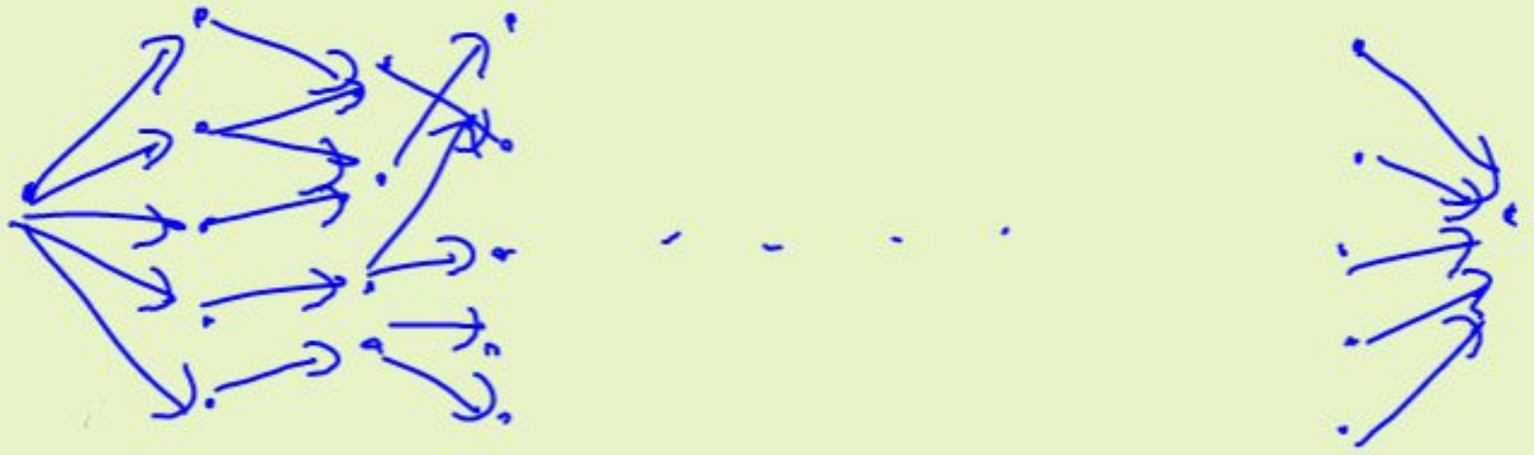
$P_{G_2} = (x, y) \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$

ABP size



size = # of vertices in graph

ABP size



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Obs: ABP A of size s $\Longrightarrow$

P_A a projection of $IMM_{s,s}$

Complexity classes

$(P_n(x_1, \ldots, x_n))_{n \geq 1}$ – a family of polys.

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$(p_n)_{n \geq 1}$ has polynomial-sized circuits if Ckt-size$(p_n) \leq n^{O(1)}$.

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VBP - poly-sized ABPs

VP - poly-sized circuits

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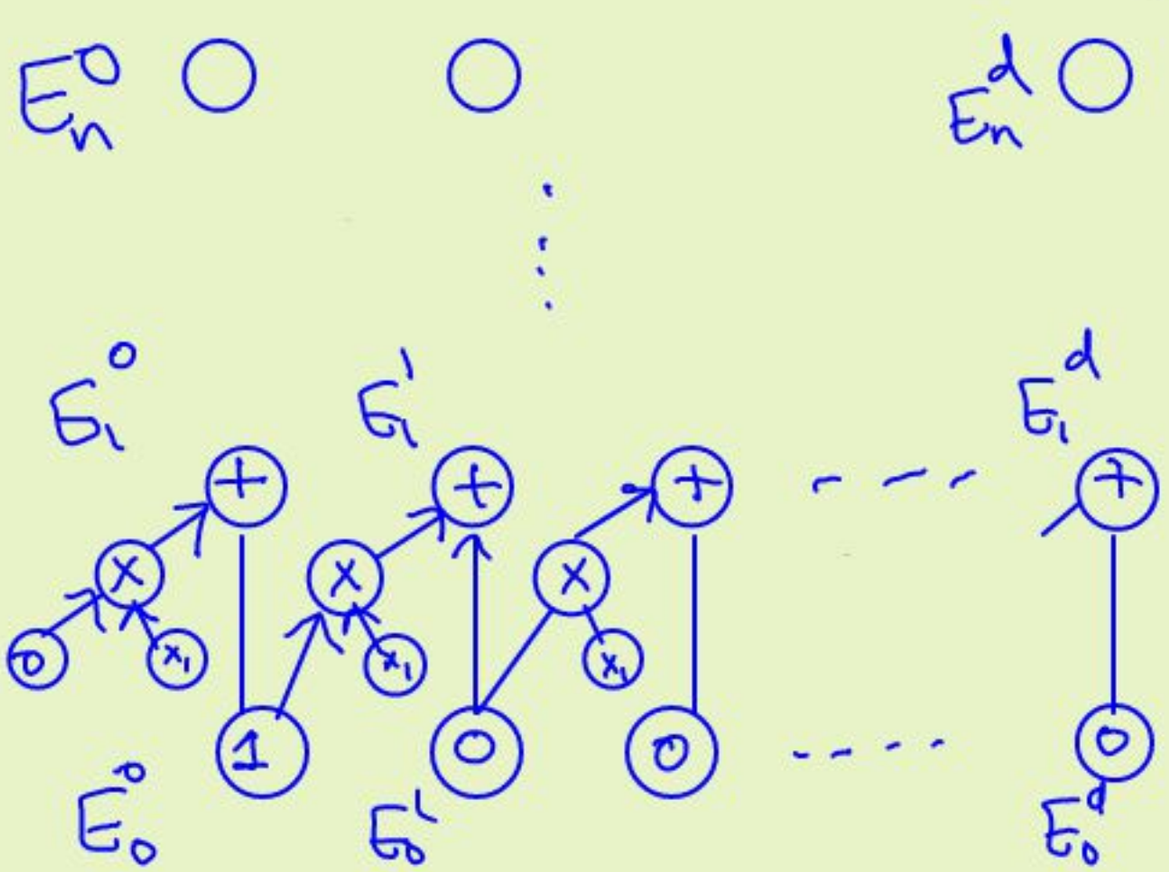

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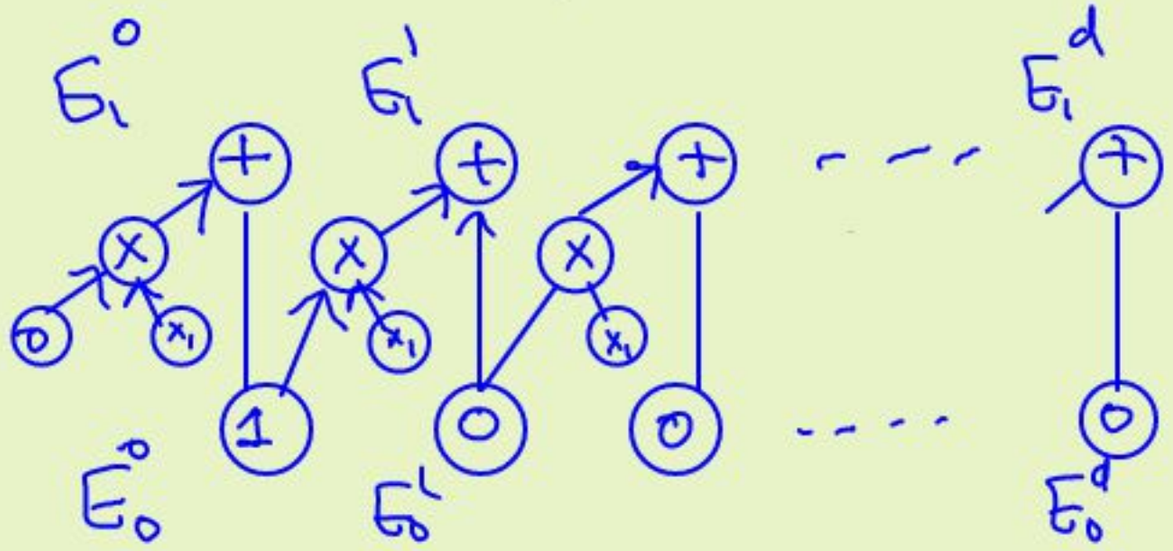
Size $\leq 4nd$
$= O(n^2)$



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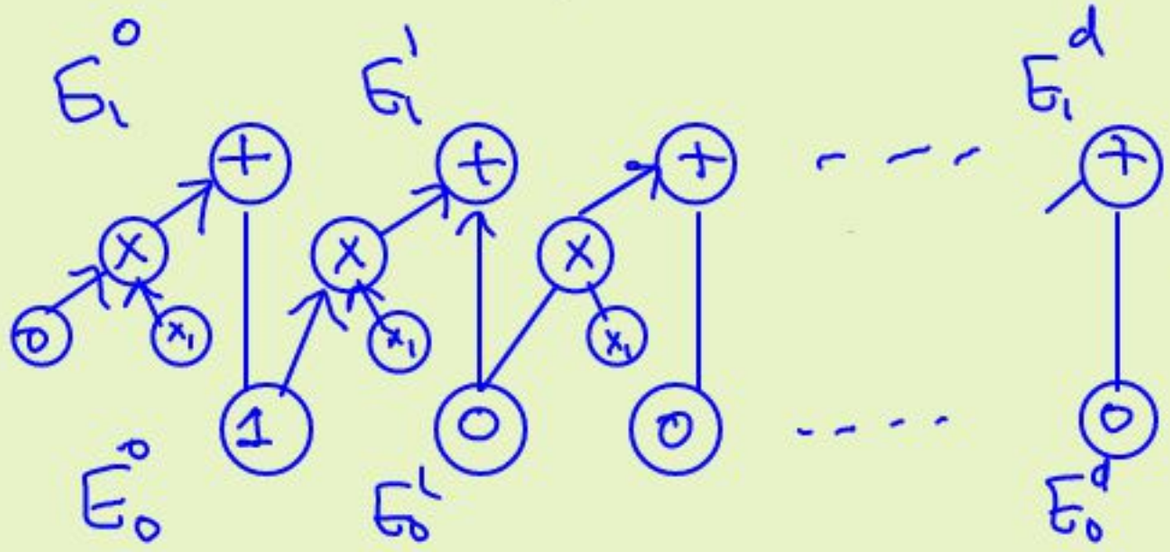

Skew circuit
- every $\times$-gate has an variable or constant input.

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$$(E_n^d, E_n^{d-1}, \dots, E_n^0) = (E_{n-1}^d, \dots, E_{n-1}^0) \begin{pmatrix} 1 & & & & \\ x_n & 1 & & & \\ & x_n & \ddots & & \\ & & \ddots & 1 & \\ & & & x_n & 1 \end{pmatrix}$$

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Exercise: Draw an ABP for E_n^d!

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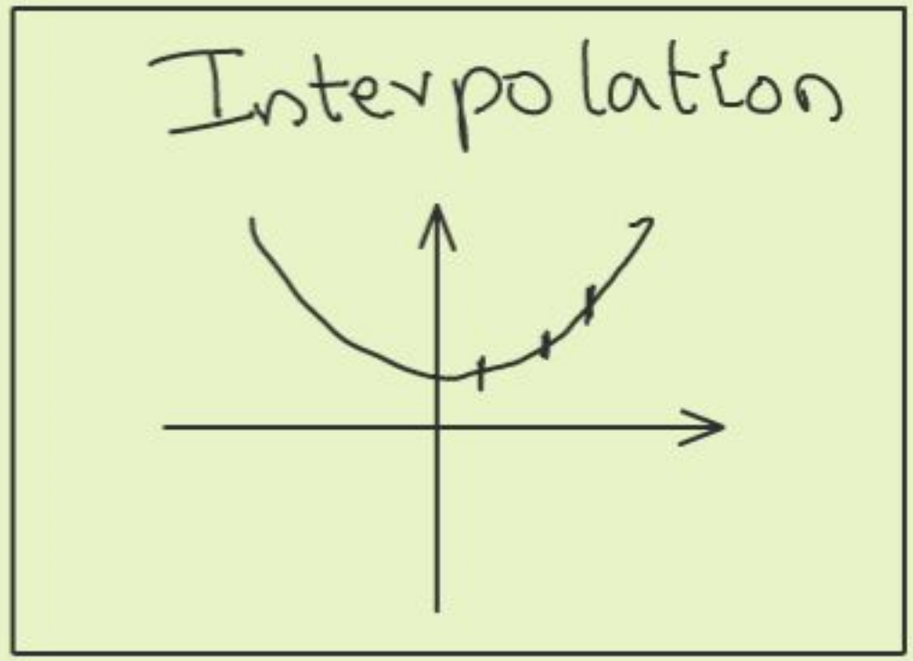
Thm: $E_n^d \in VF$
[Ben-Or]



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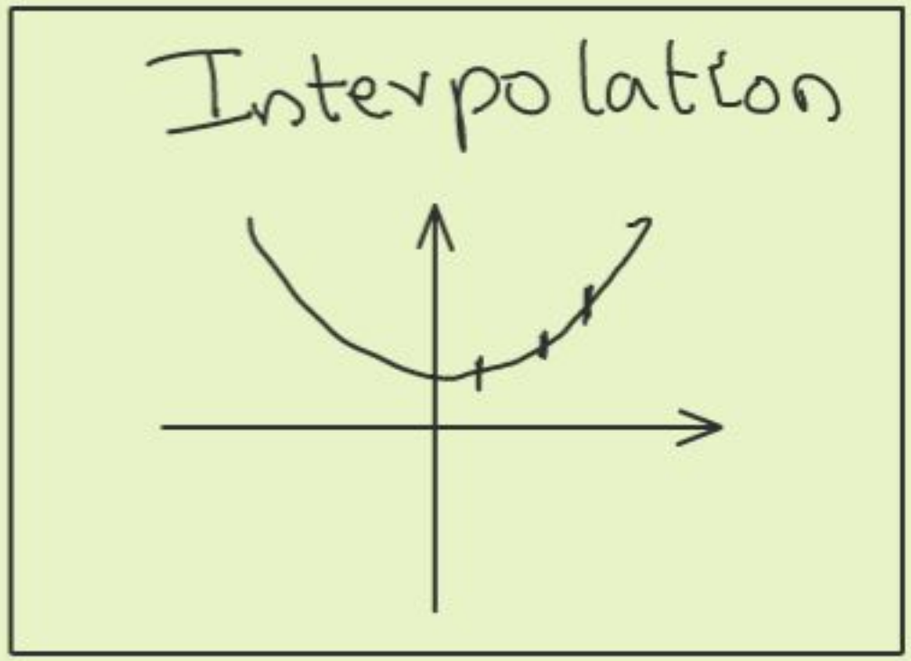
Pf: $\sum_{j=0}^{n} E_n^j \cdot t^j =$



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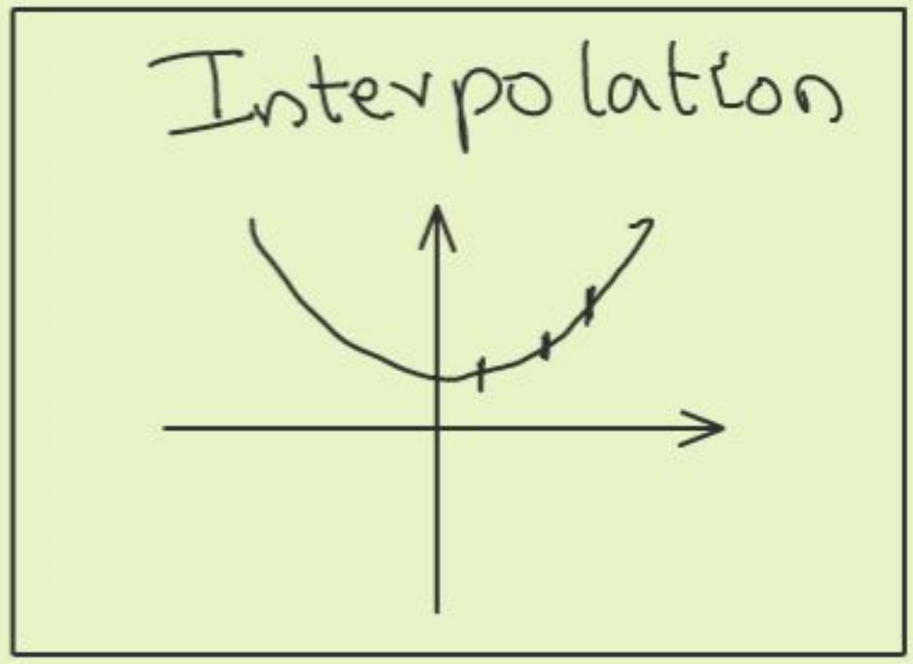

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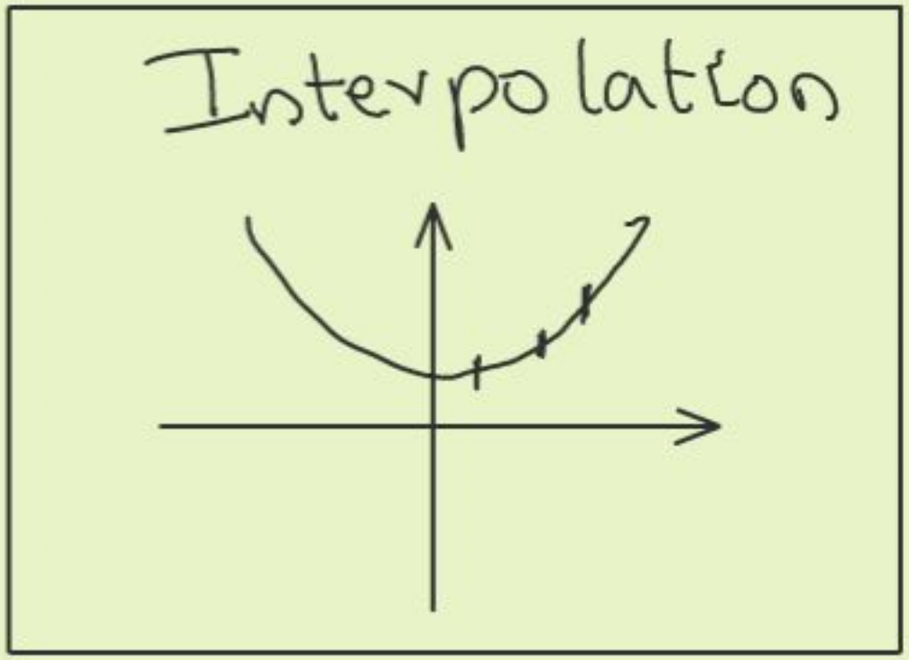
↳ formula of size $O(n)$!



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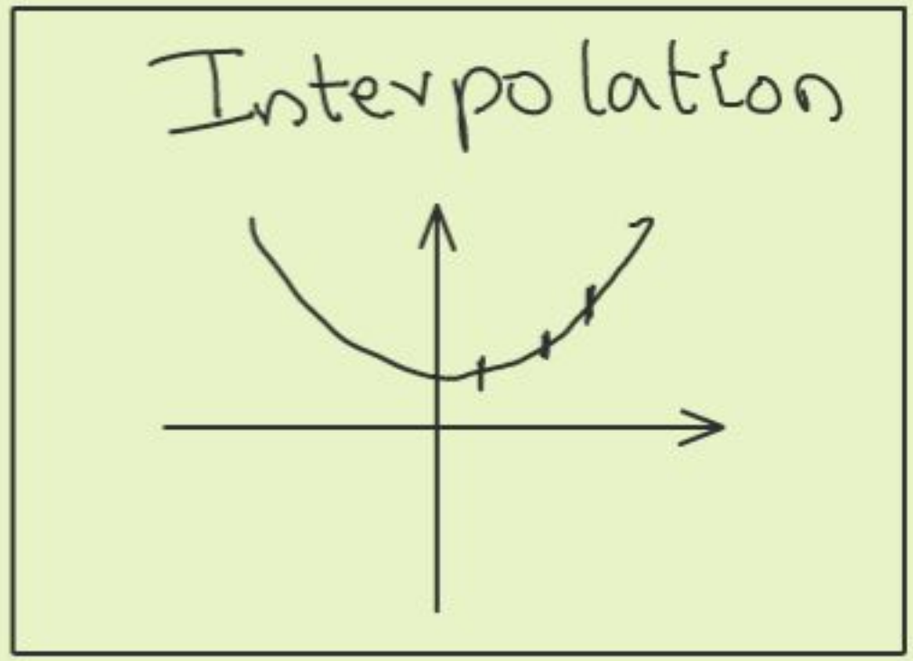

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$= \sum_{j=0}^{n} \beta_j Q(\alpha_j)$

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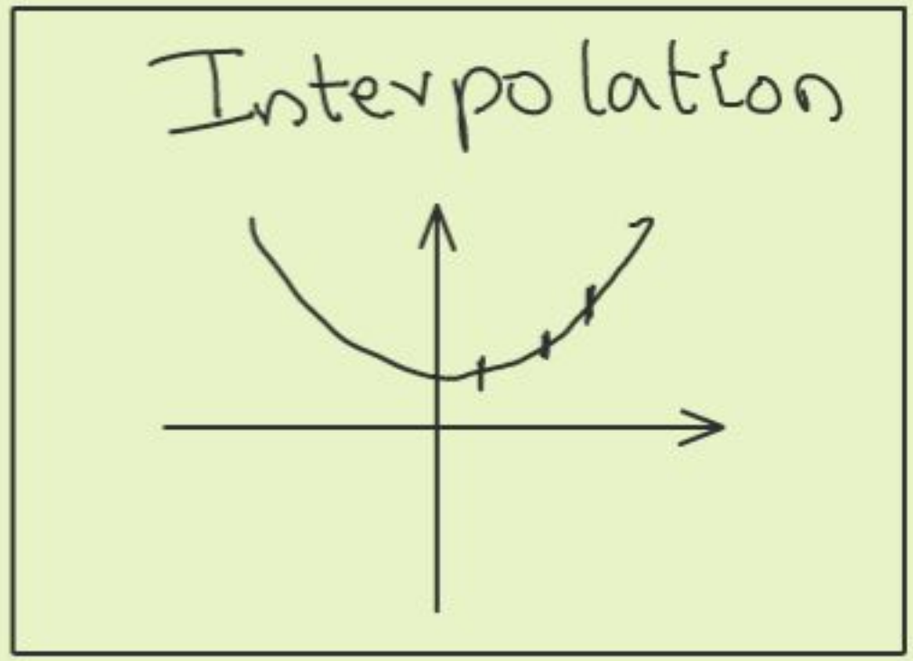

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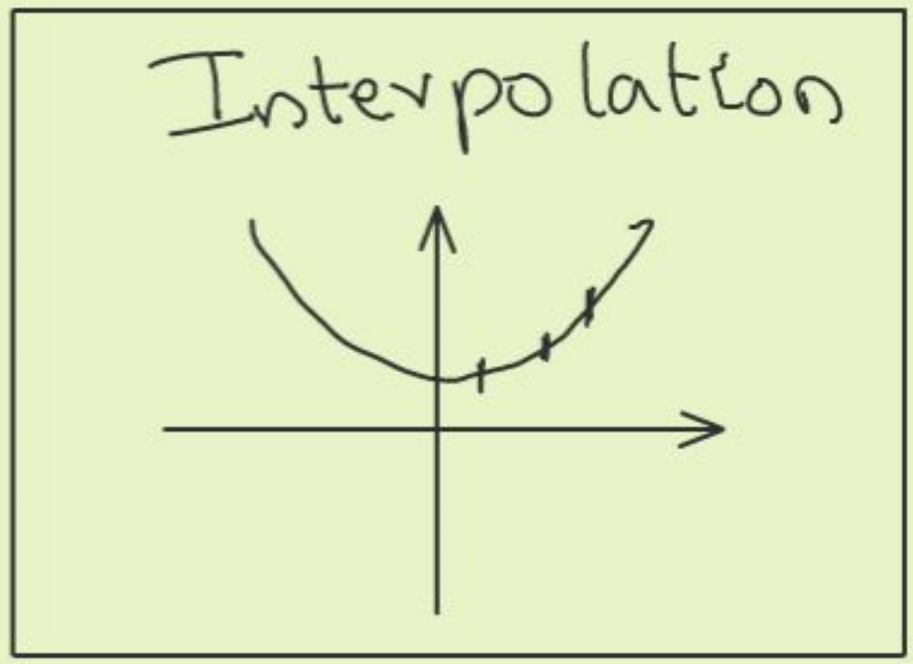

Complexity classes

$(p_n(x_1, \ldots, x_n))_{n \geq 1}$ - a family of polys.

VF - polynomial families that have poly-sized formulas

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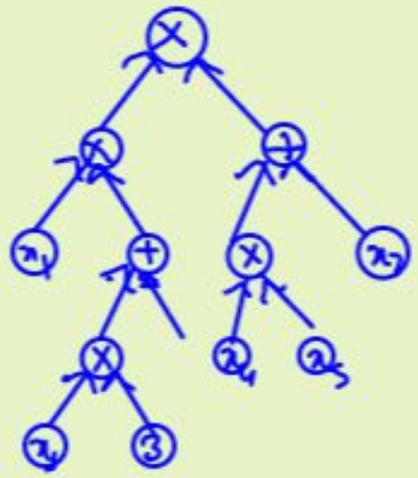

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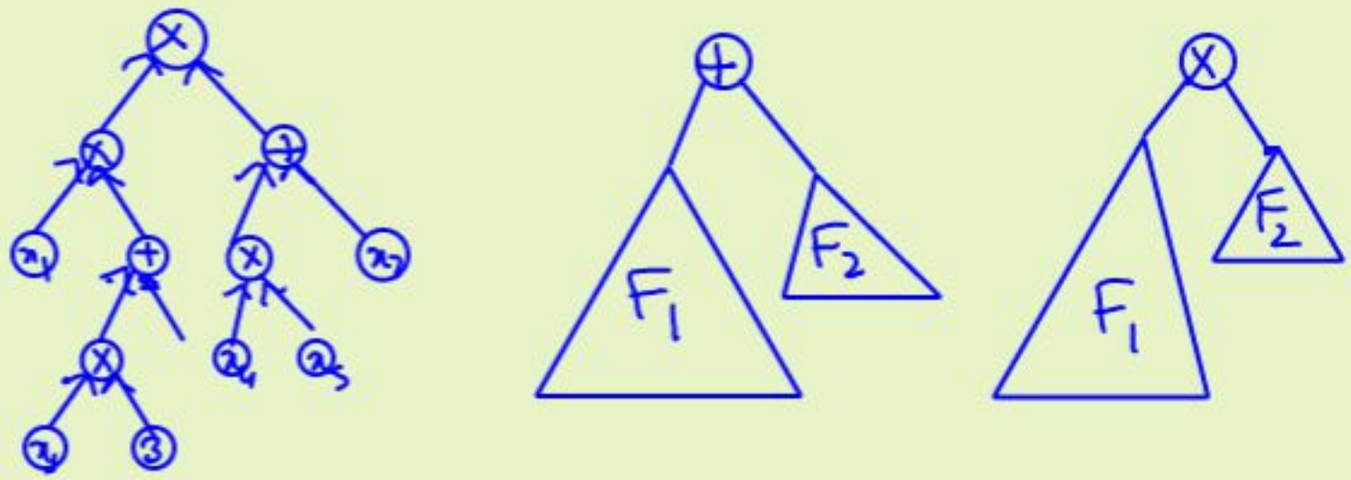

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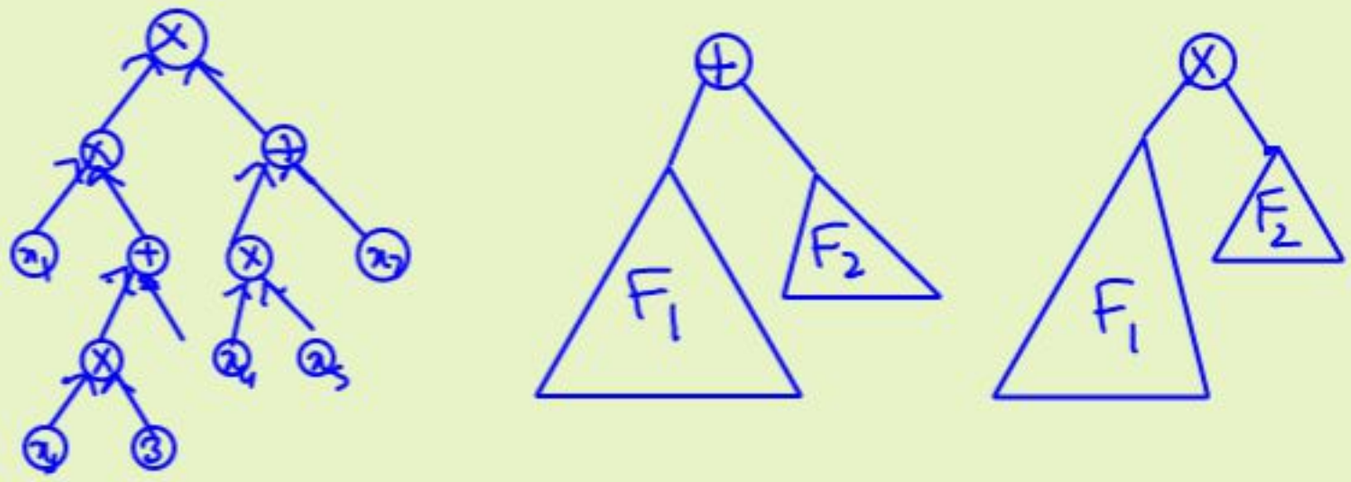

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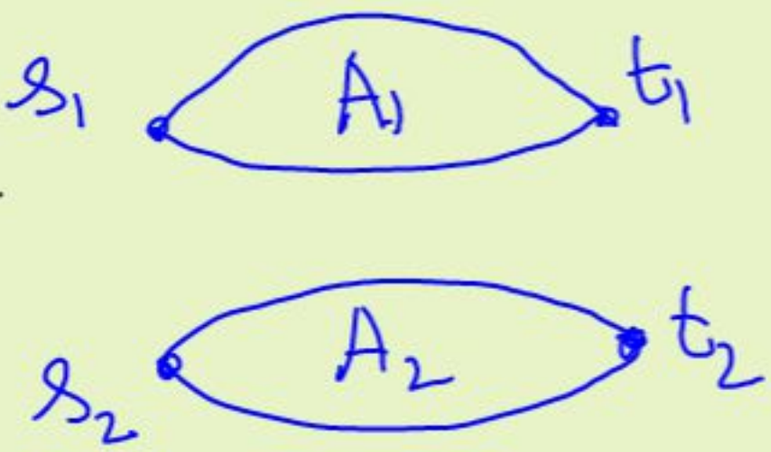

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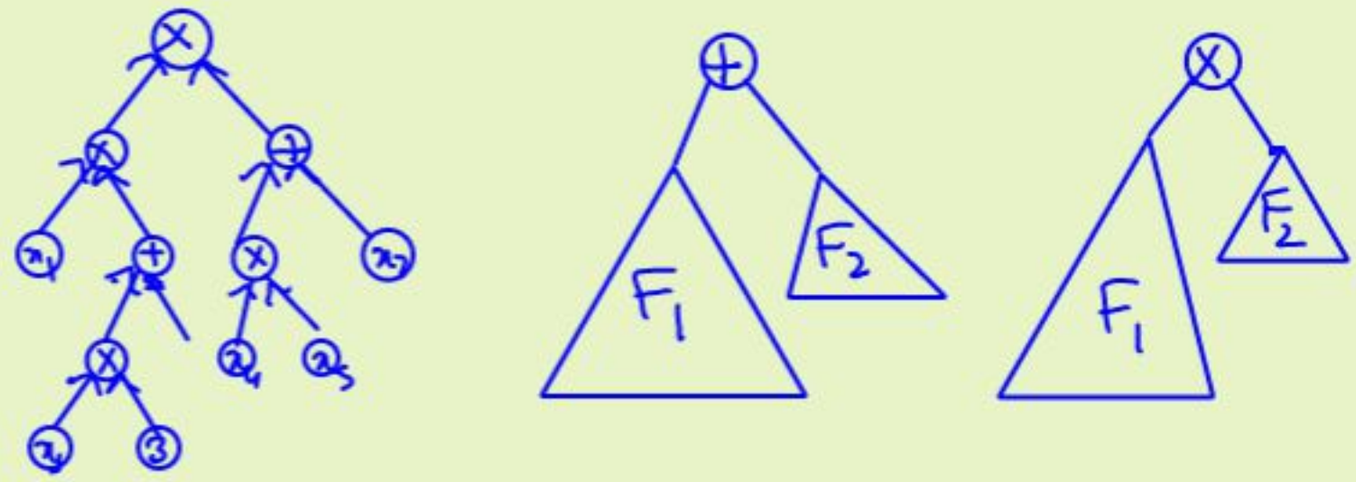

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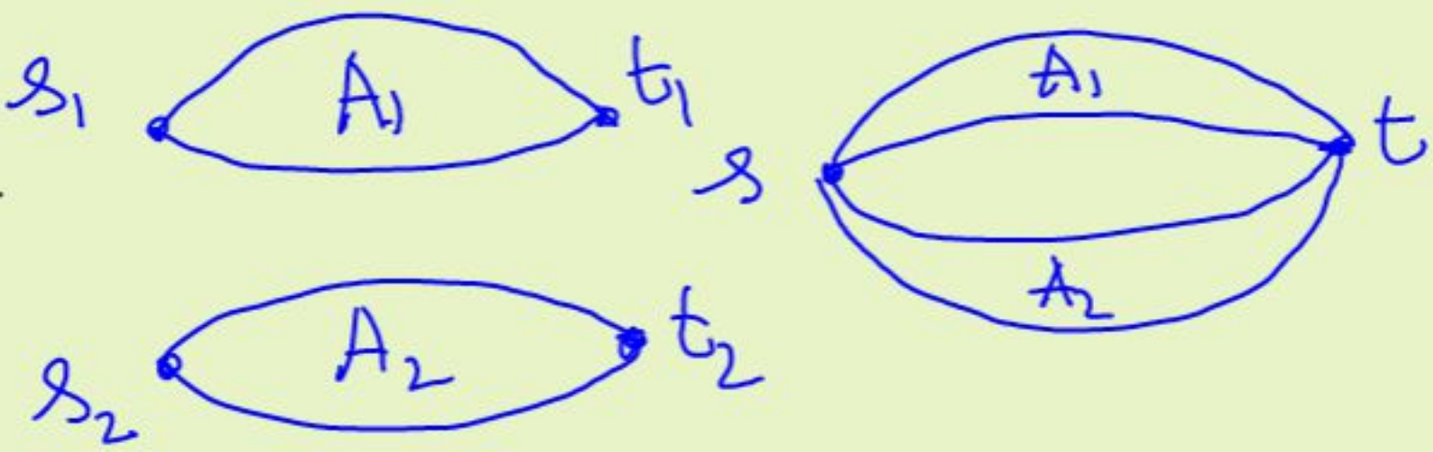

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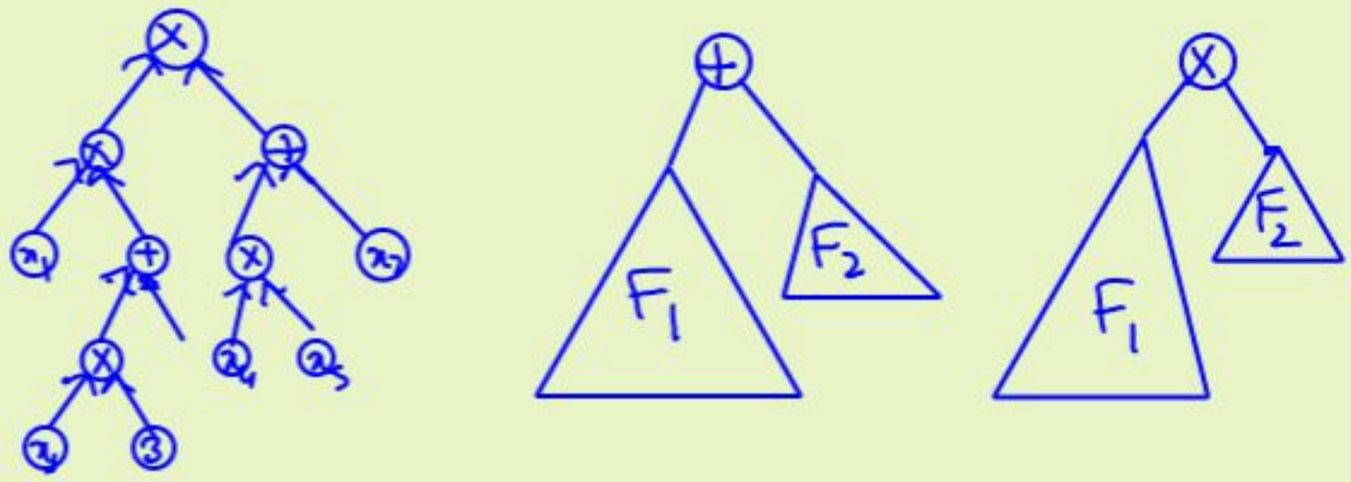

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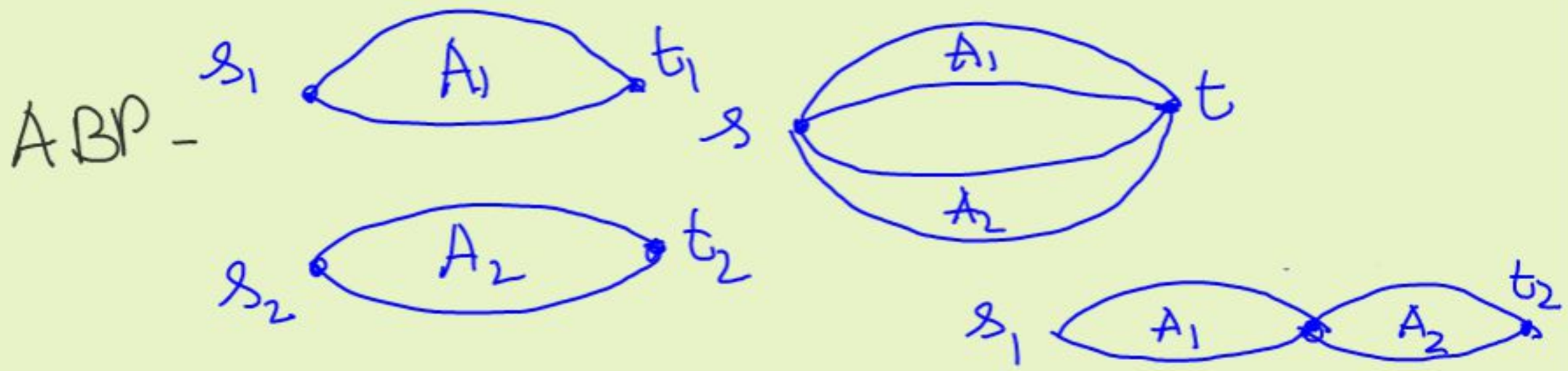

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→ linear substitution in C' gives C.

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→ (P_n) & (Q_m)

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→ p-Projections: $m(n) \leq n^{O(1)}$

→ VE, VBP, VP closed under p-projections.

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Completeness

$\mathcal{C}$ - class of polynomial sequences

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Eg : $IMM_{n,n}$ & det_n are VBP-complete.

$VF = VBP$ iff $IMM_{n,n} \in VF$ iff $det_n \in VF$

Lower bounds question

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Thm [HY]: For any n, d, $\exists p_n \in \mathbb{F}[x_1, \ldots x_n]$

$\rightarrow \deg(p_n) \leq d$, $\rightarrow$ Coeffs$(p) \in \{0,1\}$

$\rightarrow$ Any ckt. for p_n is "large."
[essentially trivial]

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Special case – $(p_n(x_1,\dots,x_n))_{n\geq 1}$ explicit if

$n \longrightarrow$ [Poly-time algo] $\longrightarrow$ Coeff$(\bar{x}^{\bar{e}})$ in p_n

$e_1,\dots,e_n \geq 0$

$\sum_j e_j \leq d$

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Ex: Show $per_m \in VNP$ using above definition

Hint: $r = m^2$ & $Q(x,M) = \begin{cases} 0 & \text{if } M \text{ not a permutation matrix} \\ \sigma^{th} \text{ term in } per_m & M = M_\sigma (\sigma \in S_m) \end{cases}$

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Known: per_n a projection of det_{2^n}
[Grenet]

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1. Formally easier than the (non-uniform) P vs. NP question.

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2. Polynomial Identity Testing

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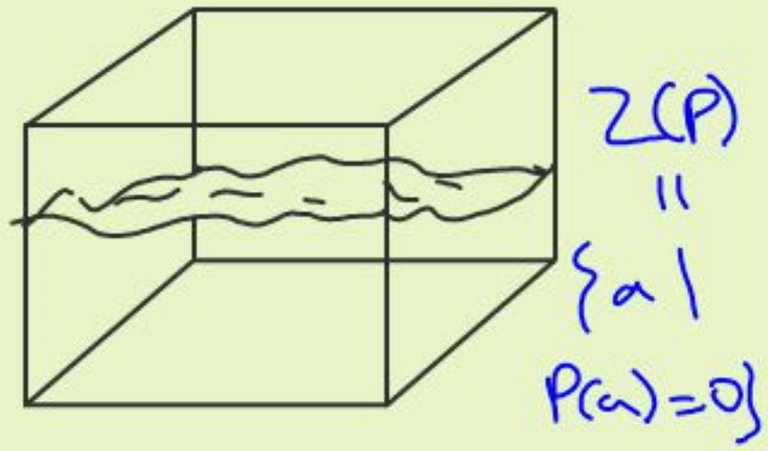
Obvious algm: Expand and check!

Takes time $\binom{n+d}{d}$. Want poly-time.

Randomized Algorithm

Randomized Algorithm

Non-zero P defines a hyper-surface in $\mathbb{F}^n$. "Most points" not on hypersurface.



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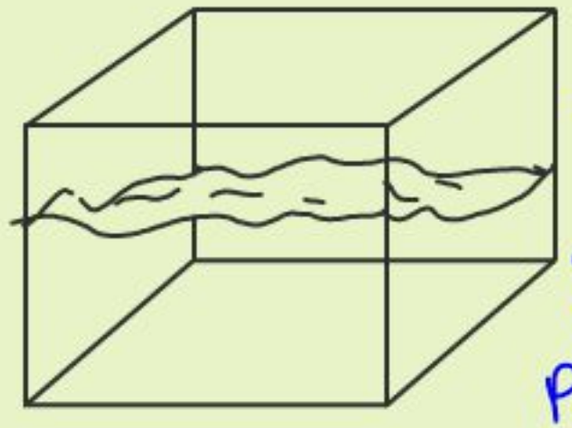

$Z(P)$ = $\{a \mid P(a)=0\}$

Polynomial Identity Lemma [Schwarz, Zippel, DeMillo, Lipton, Ore]

$$\frac{|Z(P) \cap S^n|}{|S|^n} \leq \frac{\deg(P)}{|S|}$$

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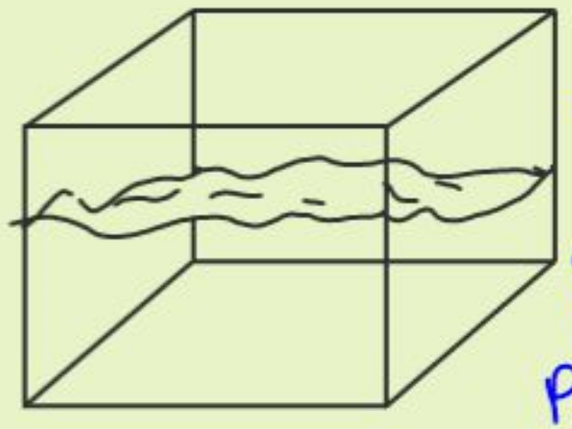

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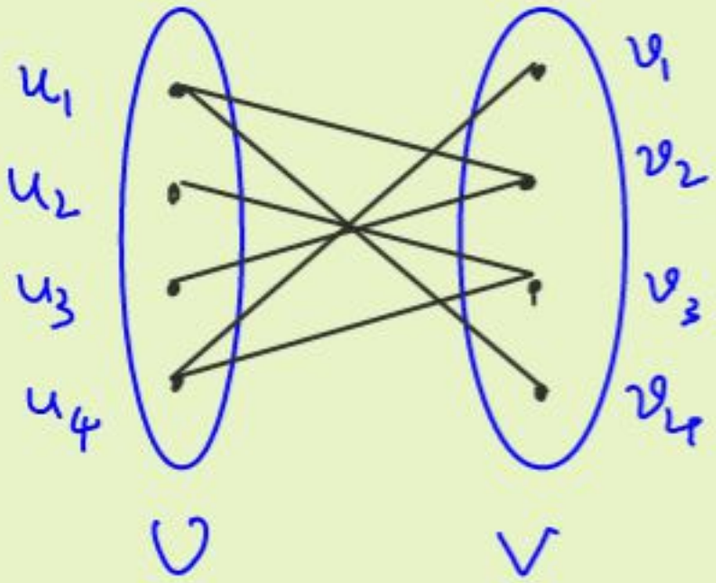
Algm. ① Fix $S \subseteq \mathbb{F}$ s.t. $|S| \geq 10d$.

② Choose $a \in_R S^n$ & test if $P(a)=0$.

Matchings



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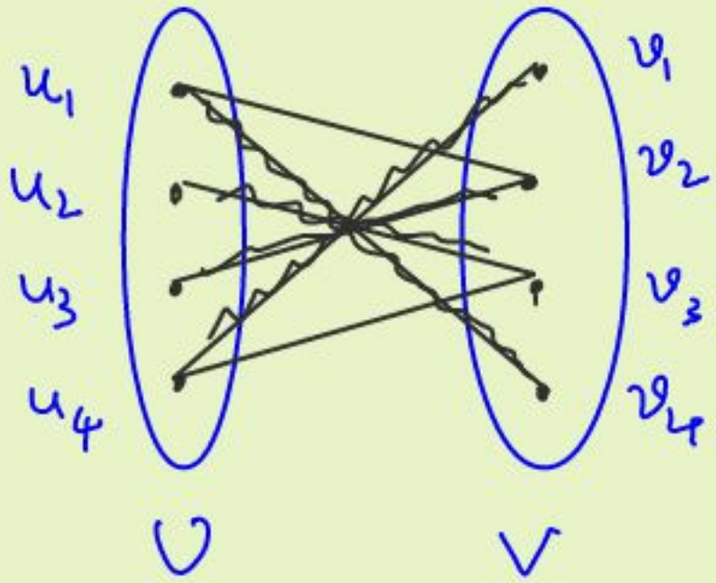

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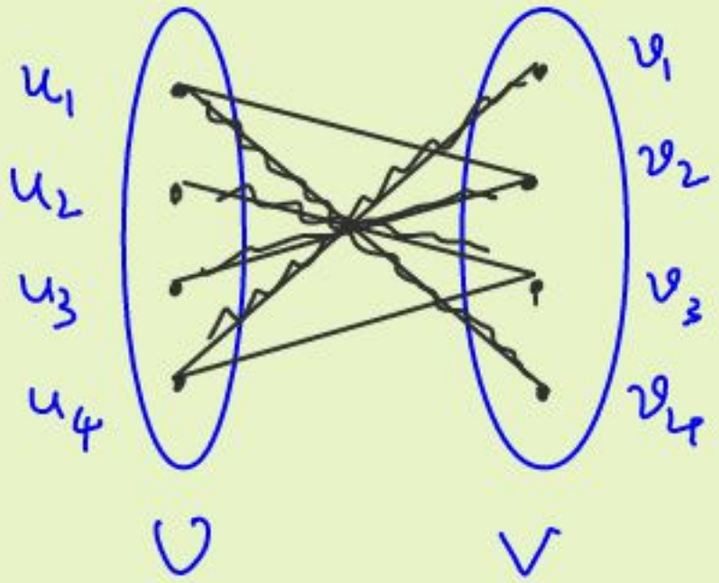

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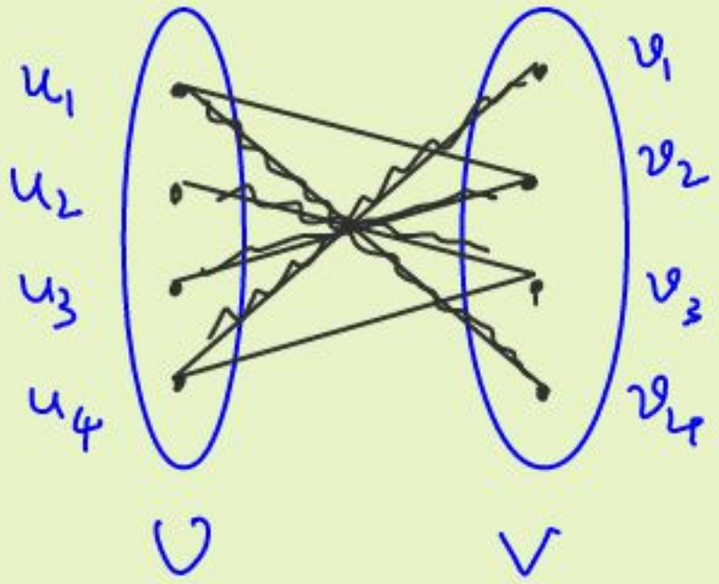

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Lovász: Translation to PIT.

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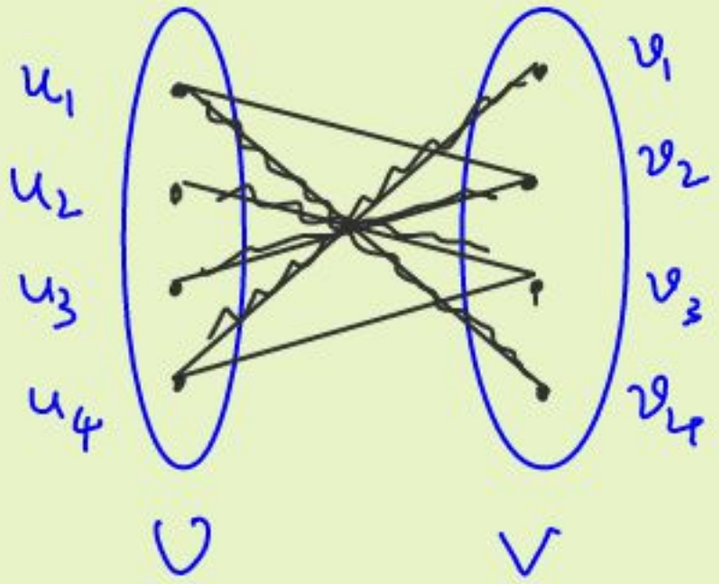

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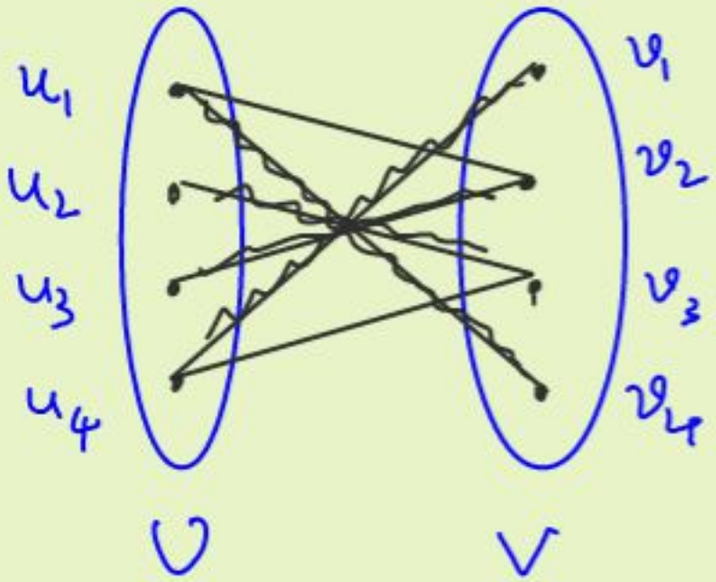

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$\det(E_G) \neq 0$ iff G has a P.M.

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"Hardness vs. Randomness": Blum-Micali, Yao, Nisan-Wigderson

Much more work since: Dvir-Shpilka-Yehudayoff, Oliveira, Chou-Kumar-Solomon, Guo-Kumar-Saptharishi-Solomon.

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More known for restricted models

Homogeneous, Monotone, Multilinear,

Non-commutative, $\Sigma\Pi\Sigma$, $\Sigma\Lambda\Sigma$, ...

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Thanks!