GIT AND MOMENT MAPS À LA NESS

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For GIT, the best account is still P. E. Newstead: *Introduction to Moduli Problems and Orbit Spaces*, Tata Institute of Fundamental Research Publications, Volume: 17; 2011

For much more on symplectic geometry, GIT and the topology of quotients, please see: F. C. Kirwan, *Cohomology of Quotients in Complex and Algebraic Geometry,* Mathematical Notes 31, Princeton University Press, Princeton N. J., 1984.

We will mostly follow the wonderful paper: Linda Ness, *A stratification of the null cone via the moment map*, American Journal of Mathematics 106(6): 1281-1329, which is best suited to our needs.

1. Basics

We work over the complex numbers $\mathbb{C}$. All group actions are on the left. Given a G-set X, we denote by X^G the set of fixed points. Given a vector space V, we let $\hat{V}$ denote its dual.

1.1. Algebraic actions of algebraic groups.

An (algebraic) action of an affine algebraic group G on an affine variety X is in particular a morphism $G \times X \to X$, and hence encoded in the induced map $\mathbb{C}[X] \to \mathbb{C}[G \times X] = \mathbb{C}[G] \otimes \mathbb{C}[X]$. The tensor product on the right is the algebraic tensor product, and this has the following important consequences:

(1) Given a regular function p on X, its G-translates span a finite-dimensional vector space of functions.

(2) Given an affine G-variety X, there exists a closed G-equivariant embedding $X \hookrightarrow V$ into a finite-dimensional vector space V with a linear G-action.

Proof. (1) Let p be a regular function, and suppose

$$p \mapsto \sum_{\alpha} q_\alpha \otimes p_\alpha$$

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By definition, if $g \in G$ and $x \in X$, we have

$$p(g^{-1}x) = \sum_\alpha q_\alpha(g) p_\alpha(x)$$

so the G-translates of p are contained in the span of the p_α.

(2) Choose a (finite) set of regular functions that generate $\mathbb{C}[X]$, then enlarge them to a finite set whose linear span is invariant under G. $\square$

From now on, we will only consider algebraic actions of affine algebraic groups on quasi-projective (usually affine or projective) varieties.

A linear action of G on a finite-dimensional vector space is also called a *representation*.

Let an affine algebraic group G act linearly on a f.d. vector space V. It acts on the algebra of polynomial functions $\mathbb{C}[V] = S^*\hat{V} = \mathbb{C} \oplus \{\oplus_{k=1,\dots} S^k\hat{V}\}$ by:

$$(p, g) \mapsto p^g, \quad p^g(x) = p(g^{-1}x)$$

Since the linear action preserves degrees, we have

$$\mathbb{C}[V]^G = \mathbb{C} \oplus \{\oplus_{k=1,\dots} [S^k\hat{V}]^G\}$$

In other words, if $p = p_0 + \dots + p_k$ is a G-invariant polynomial, so are its homogeneous terms $p_0, \dots, p_k$ of each degree. Let $\mathcal{I}$ denote the ideal generated by all invariant polynomials with zero constant term, i.e.,

$$\mathcal{I} = < \oplus_{k=1,\dots} [S^k\hat{V}]^G >$$

By the Hilbert Basis Theorem, there exist finitely many homogeneous invariant polynomials $q_1, \dots, q_r$, of strictly positive degrees $d_1, \dots, d_r$ that generate $\mathcal{I}$. The subscheme defined by $\mathcal{I}$

$$\mathcal{N} = \{x \in V | q(x) = 0 \forall q \in \mathcal{I}\} = \{x \in V | q_i(x) = 0 \ i = 1, \dots, r\}$$

is the (affine) *null-cone*. For future use, note that the null-cone is indeed a *conical* subvariety, i.e., a closed subvariety of the vector space V that is invariant under scalar multiplication. (This is because it is defined by the vanishing of homogeneous polynomials.)

The extreme cases are when G acts trivially and every polynomial is invariant, in which case, the null-cone is the origin 0_V or when the only invariant polynomials are the constants, in which case the null-cone is V itself. Excluding these cases, we see that

$$0_V \subsetneq \mathcal{N} \subsetneq V$$

and $\mathcal{N}$ is a closed conical (that is, stable under scalar multiplication) subscheme. In general $\mathcal{N}$ need not be reduced[1] or irreducible. (We should therefore use the word scheme, since the term "variety" is often restricted

[1] Consider the action $z \mapsto \pm z$ of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}$.

to reduced schemes. For the most part, we can and will ignore scheme-theoretic subtleties.)

We note the following result of Borel. (I am quoting conveniently from stackexchange!):

If G is an algebraic group acts on an algebraic variety X, then each orbit Gx is a smooth algebraic variety, open in its Zariski closure $\overline{Gx}$. The set $\overline{Gx} \smallsetminus Gx$ is a union of orbits of strictly lower dimension. Each orbit Gx is a constructible set, hence the Zariski closure $\overline{Gx}$ coincides with the closure in the standard Euclidean topology.

Another (deeper) result is Matsushima's theorem: if G is reductive, G/H is affine iff H is reductive.

1.2. **Reductive=linearly reductive groups.** We will deal exclusively with *reductive* groups. This is a affine algebraic group such that any maximal compact subgroup K is Zariski-dense. Examples are:

(1) The group $\mathbb{C}^*$ of nonzero complex numbers, with (in this case unique) maximal compact S^1, the group of complex numbers of modulus 1.

(2) More generally, complex tori:

$$\underbrace{\mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{n \textit{ times}}$$

with (also unique) maximal compact subgroup

$$\underbrace{S^1 \times \cdots \times S^1}_{n \textit{ times}}$$

(3) Finite groups.

(4) Simple Lie groups, say $SL(2,\mathbb{C})$, with maximal compact $SU(2)$ (or any of its conjugates.).

(5) Products of the above, and quotients by finite normal subgroups.

A reductive group is the same as a *linearly reductive* group, i.e., one for which every representation is completely reducible. In particular, given a linear action of such a G on a finite-dimensional vector space W, we have a unique splitting compatible with the G-action:

$$W = W^G \oplus W'$$

or equivalently, a G-equivariant projection $r_W : W \to W^G$. (Choose a maximal compact subgroup K, average any inner product on W over K to get a K-invariant inner product, and let W' be the orthogonal complement of W^G. This is K-invariant, and G-invariance (of W' *not* the inner product!) follows from the Zariski-density of K.)

If a linearly reductive G acts linearly on a vector space V, the projections

$$r_{S^k\hat{V}} : S^k\hat{V} \to (S^k\hat{V})^G, k = 0, 1, \dots,$$

together give the *Reynolds operator*

$$r : \mathbb{C}[V] \to \mathbb{C}[V]^G$$

which satisfies

(1) $r(q) = q$ for every invariant polynomial q, and
(2) $r(qp) = qr(p)$ if q is invariant.

We will use the following repeatedly. Let G be a linearly reductive group.

(1) A Reynolds operator $r : \mathbb{C}[X] \to \mathbb{C}[X]^G$ exists for any affine G-variety X.
(2) If X is an affine G-variety and Y a closed G-invariant subvariety, the restriction map

$$\mathbb{C}[X] \to \mathbb{C}[Y]$$

has a G-equivariant splitting. In particular a G-invariant regular function on Y has a G-invariant extension to X. That is, the map

$$\mathbb{C}[X]^G \to \mathbb{C}[Y]^G$$

is onto.

Let us now consider a linear action of a linearly reductive group G on a vector space V. Using the Reynolds operator, one can prove:

Theorem 1.1. *Let $q_1, \dots, q_r$ be homogeneous invariant polynomials that generate the ideal $\mathcal{I}$, or equivalentlly, define the null-cone $\mathcal{N}$. Then they generate the invariant ring $\mathbb{C}[V]^G$. In particular, the invariant ring is finitely-generated.*

Proof. Let P be a homogeneous invariant polynomial of minimal degree that is not in the subalgebra generated by $q_1, \dots, q_r$. We have

$$P = \sum_i p_i q_i$$

where $deg\ p_i = deg\ P - deg\ q_i < deg\ P$. Applying the Reynolds operator, we get

$$P = \sum_i r(p_i) q_i$$

Since $r(p_i)$ are invariant and of degree $< deg\ P$, they are in the subalgebra generated by the q_i; contradiction. (This argument might read better formulated as an induction.) □

Suppose given an action of a linearly reductive group G on an affine variety[2] X. One can bootstrap our previous claims to yield: *the invariant ring* $\mathbb{C}[X]^G$ *is finitely-generated.*

Let $X//G$ denote the affine variety with coordinate ring $\mathbb{C}[X]^G$. That is to say, $\mathbb{C}[X//G] = \mathbb{C}[X]^G$. The inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$ yields a map (*a priori* only dominant, i.e., with Zariski-dense image) $\pi : X \to X//G$. Concretely, if $\mathbb{C}[X]^G$ is generated by invariant regular functions $q_1, \dots, q_N$, the map

$$\tilde{\pi} : X \to \mathbb{C}^N, \quad x \mapsto (q_1(x), \dots, q_N(x))$$

has image the closed subvariety $X//G$ defined by equations expressing algebraic identities between the q_i. We write

$$X \underset{\pi}{\longrightarrow} X//G \underset{inclusion}{\hookrightarrow} \mathbb{C}^N$$

The choice of generators is not unique; the dimension N and the specific way $X//G$ is expressed as an affine sub-variety depend on this choice.

In fact, the morphism π has many nice properties. To make a proper definition, consider a morphism $\pi : X \to Y$ of quasi-projective varieties, with an algebraic group G acting on X. We say that π is a *good quotient* if the following hold.

(1) The map π is onto. It is G-equivariant with G acting trivially on Y, which is a fancy way of saying: π is constant on G-orbits (i.e., $\pi(gx) = \pi(x) \forall x \in X,\ g \in G$.)

(2) Any G-equivariant map $X \to Y'$, with G acting trivially on Y', factors uniquely through π. (We say that π is a "categorical" quotient.)

(3) The map π is an affine morphism. That is, given any affine open $U \subset Y$, the morphism $\pi|_{\tilde{U}} : \tilde{U} \to U$ is a morphism.of affine varieties, where $\tilde{U} \subset X$ denotes the inverse image of U.

(4) Given any affine open $U \subset Y$, we have $\mathbb{C}[U] = \mathbb{C}[\pi^{-1}(U)]^G$; equivalently, the natural map $\mathbb{C}[U] \to \mathbb{C}[\pi^{-1}(U)]^G$ is an isomorphism. In sheaf-theoretic terms, $\mathcal{O}_Y$ is the invariant direct image of $\mathcal{O}_X$.

(5) Given a closed G-invariant subvariety $X' \subset X$, its image $\pi(X')$ is closed in Y.

(6) Given disjoint closed G-invariant subvarieties X', X'' in X, the images $\pi(X')$ and $\pi(X')$ are disjoint.

Note that the definition of good quotient does not require G to be reductive. Because of possible subtleties, *we will assume that when we talk of*

[2] so we are generalising the situation from the previous one, in that the variety in question need not be a vector space, and the action need not be linear.

good quotients, G is reductive and we are working over $\mathbb{C}$ unless otherwise specified explicitly.

The variety $X//G$, together with the morphism $\pi : X \to X//G$ is a good quotient, and called an *affine geometric invariant theory (GIT) quotient.*

Note the following facts:

(1) if $X' \subset X$ is a closed G-invariant subavariety, the functorial map $X'//G \to \pi(X')$ is an isomorphism of affine varieties.

(2) $\pi(x) = \pi(x')$ iff the G-orbit closures $\overline{Gx}$ and $\overline{Gx'}$ intersect.

(3) Given $y \in X//G$, the inverse image $\pi^{-1}(y)$ is a union of G-orbits, and contains precisely one closed G-orbit. So *the points of $X//G$ are in bijection with closed G-orbits.*

Let $y \in Y$, and consider $\pi^{-1}(y)$. The possibilities are:

(1) $\pi^{-1}(y)$ consists of only one orbit, with its dimension equal to that of G, in which case, G acts with finite isotropy at any of the points along the orbit.

(2) $\pi^{-1}(y)$ consists of only one orbit, with its dimension strictly less than that of G, in which case, G acts with positive-dimensional isotropy at any of the points along the orbit.

(3) $\pi^{-1}(y)$ consists of more than one orbit, in which case, it contains precisely one closed orbit, whose dimension is minimal among all orbits in $\pi^{-1}(y)$. The closure of any orbit in $\pi^{-1}(y)$ must contain the closed orbit.

An example of a good quotient which is not an affine GIT quotient is the familiar construction of the projective space $\mathbb{P}(V)$ of one-dimensional subspaces of a vector space V. The map $V \setminus \{0_V\} \to \mathbb{P}(V)$ is a good quotient by a free action of $\mathbb{C}^*$. In fact the points of $\mathbb{P}(V)$ are in bijection with $\mathbb{C}^*$ orbits – all orbits are closed – so that this is a "geometric" quotient.

An affine (and by extension good) GIT quotient $\pi : X \to X//G$ (and by extension a good quotient) inherits many properties from X. You can check that If X is reduced so is Y, and if X is normal, so is Y. Non-singularity is rarely preserved, but the property of having rational singularities is (Boutot).

<u>Warning regarding positive characteristics</u>. In positive characteristic we need to define reductive groups differently. Very few groups are linearly reductive (although k^* itself is), but the notion of "geometric reductivity" and Haboush's resolution of Mumford's conjecture (that reductive groups are geometrically reductive) come to our rescue, and much of the theory goes through. Some care has to be taken. For example, it is no longer always

true that $X'//G$ sits inside $X//G$ if $X' \hookrightarrow X$. (This is because an invariant function on a closed invariant subvariety need not lift to an invariant function on the ambient variety.)

1.3. **Action on a vector space; null-cone.** Consider a linear action of a linearly reductive group G on a vector space V. Let $\pi : V \to V//G$ be the GIT quotient. Recall that $V//G$ is the affine variety with coordinate ring $\mathbb{C}[V]^G$

For future use, note that both $\mathbb{C}[V]$ and $\mathbb{C}[V]^G$ are graded $\mathbb{C}$-algebras. The first of these is in fact generated by its degree one homogeneous component $S^1\hat{V} = \hat{V}$. If G is linearly reductive, the invariant ring $\mathbb{C}[V]^G$ *is* finitely generated, but it need not be generated by homogeneous invariant polynomials of degree one, or even of equal degree.

Let us introduce the notation $R = \mathbb{C}[G]$, and denote by R_k the k^{th} graded piece, i.e., $R_k = S^k\hat{V}$. The invariant ring is $R^G = \sum_k R_k^G$. Let $R_{>0}, R_{>0}^G$ denote the "irrelevant maximal ideals":

$$R_{>0} = \oplus_{k>0} R_k$$

$$R_{>0}^G = \oplus_{k>0} R_k^G = R^G \cap R_{>0}$$

Note that $R_{>0}$ is the (maximal) ideal defining the origin 0_V. Let $\nu \in V//G$ denote the point defined by $R_{>0}^G$. This is the scheme-theoretic image of 0_V. We have $\mathcal{N} = \pi^{-1}(\nu)$ (scheme-theoretic inverse image).

1.4. **Projective quotients.** Let $\mathbb{C}^*$ act on V by $(t, \vec{v}) \mapsto t\vec{v}$ (scalar multiplication of the vector $x \in V$ by the scalar t. The induced action on $S^k\hat{V}$ is by $q \mapsto t^{-k}q$. This restricts to the invariant subring, and induces an action of $\mathbb{C}^*$ on the quotient $V//G$ commuting with π:

$$\pi(tx) = t \cdot \pi(x)$$

Let $\mathbb{P}(V//G)$ denote the *Proj* (variety rather than scheme) of the graded ring R^G. (Note that $Proj(R) = \mathbb{P}(V)$.) There is a natural map $V//G \smallsetminus \{\nu\} \to \mathbb{P}(V//G)$. This is in fact a good quotient by the action of $\mathbb{C}^*$. We have therefore a commutative diagram of maps:

$$\begin{tikzcd} V \smallsetminus \mathcal{N} \arrow[r] \arrow[d, "\pi|_{V\smallsetminus\mathcal{N}}"] & \mathbb{P}(V)^{ss} \equiv \mathbb{P}(V) \smallsetminus \mathbb{P}(\mathcal{N}) \arrow[d, "\pi_{\mathbb{P}}"] \\ V//G \smallsetminus \{\nu\} \arrow[r] & \mathbb{P}(V//G) \end{tikzcd}$$

That $\pi_{\mathbb{P}}$ is a good quotient is not entirely obvious, but true. We have snuck in the all-important definition and notation, that of *semistable* points of $\mathbb{P}(V)$ under the action of a reductive group G, and the GIT quotient $\mathbb{P}(V//G)$ of the open set $\mathbb{P}(V)^{ss}$ of semistable points. Note that $\mathbb{P}(V//G)$ is projective even though $\mathbb{P}(V)^{ss}$ is usually only quasi-projective.

(Though the above point of view is implicit in many accounts, it is systematically used in lecture notes by M.S. Narasimhan, where he uses the "Ramanathan Lemma" to justify the arguments.)

Given a nonzero vector $\vec{v} \in V$, we will let $[\vec{v}] \in \mathbb{P}(V)$ denote the corresponding one-dimensional vector space and say that $\vec{v}$ is "above" or "over" $[\vec{v}]$. Note that $[\vec{v}] = [\vec{v}']$ iff the two vectors are nonzero scalar multiples of each other. Since $\mathcal{N}$ is a conical subvariety of V, it determines a closed closed subvariety $\mathbb{P}(\mathcal{N}) \subset \mathbb{P}(V)$ is called the (projectivised) *null-cone*.

A point $[\vec{v}] \in \mathbb{P}(V)$ is said to be *semistable* if it lies in the complement of $\mathbb{P}(\mathcal{N})$. To be explicit: *a point $[\vec{v}] \in \mathbb{P}(V)$ is semistable if there is a homogeneous invariant polynomial q of positive degree which does not vanish at any point $\vec{v} \in V$ "above" $[\vec{v}]$.*

Points on the null-cone are called *unstable* by Mumford, but it is safer to use the term *non-semistable.*

We can successively generalise the situation.

(1) First, if the action of G on $\mathbb{P}(V)$ leaves invariant a closed subvariety $X \subset \mathbb{P}(V)$, we can define $X^{ss} \equiv X \cap \mathbb{P}(V)^{ss}$, and we obtain a good quotient (red arrow below)
$$\mathbb{P}(V)^{ss} \underset{closed}{\hookleftarrow} X^{ss} \rightarrow X^{ss}//G \underset{closed}{\hookrightarrow} \mathbb{P}(V//G)$$
(2) Suppose given a projective variety X with a G-action. To define the notion of semistability, we need to first choose an ample line bundle L and a lift of the G-action to L. Then one can bootstrap what went before and obtain a good quotient $X^{ss} \to X^{ss}//G$, with $X^{ss}//G$ projective. Some power of L descends to $X^{ss}//G$ and is ample there.

Let us recall the list of possibilities for the fibre $\pi^{-1}(y)$ of a good quotient:

(1) $\pi^{-1}(y)$ consists of only one orbit (which is necessarily closed), with its dimension equal to that of G, in which case, G acts with finite isotropy at any of the points along the orbit.

(2) $\pi^{-1}(y)$ consists of only one orbit (again, necessarily closed), with its dimension strictly less than that of G, in which case, G acts with positive-dimensional isotropy at any of the points along the orbit.

(3) $\pi^{-1}(y)$ consists of more than one orbit, in which case, it contains precisely one closed orbit, whose dimension is minimal among all orbits in $\pi^{-1}(y)$. The closure of any orbit in $\pi^{-1}(x)$ must contain the closed orbit.

A point $[\vec{v}] \in \mathbb{P}(V)$ is *stable* if it is semistable with its orbit closed in $\mathbb{P}(V)^{ss}$ and the group G acts freely through it, except possibly for a finite isotropy.

(This corresponds to a point on a fibre of type (1) above.) If the group G acts "effectively" (roughly speaking, the action does not factor though the action of a quotient group of smaller dimension), the typical semi-stable but not stable point of $\mathbb{P}(V)$ lies on a fibre of type (3), and occasionally of type (2). The terms stable and semistable can be extended to nonzero vectors in V: a nonzero vector $\vec{v} \in V$ is *(semi)stable)* if the corresponding point $[\vec{v}] \in \mathbb{P}(V)$ is.

1.5. **Weighted projective spaces and $\mathbb{P}(V//G)$.** To understand $\mathbb{P}(V//G)$ it is useful to introduce weighted projective spaces. Let a sequence positive integers $\mathbf{d} \equiv \{d_1 \leq d_2 \cdots \leq d_N\}$ be given. We let $\mathbb{P}^{N-1}_{\mathbf{d}}$ denote the quotient of $\mathbb{C}^N \smallsetminus \{0_{\mathbb{C}^N}\}$ by the following action of $\mathbb{C}^*$:

$$(t, (z_1, \dots, z_N)) \mapsto (t^{d_1} z_1, \dots, t^{d_N} z_N)$$

This is a (in general singular) projective variety called a weighted projective space. Standard references are notes by Dolgachev or Miles Reid.

Given G-invariant homogeneous polynomials $q_1, \dots, q_N$ (of degrees $d_1 \leq \dots, d_N$) that generate R^G, the maps π and $\pi_{\mathbb{P}}$ are induced by the maps

$$\vec{v} \in V \mapsto (q_1(\vec{v}), \dots, q_N(\vec{v})) \in \mathbb{C}^N$$

$$[\vec{v}] \in \mathbb{P}(V)^{ss} \mapsto [(q_1(\vec{v}), \dots, q_N(\vec{v}))] \in \mathbb{P}^{N-1}_{\mathbf{d}}$$

Note that since $[\vec{v}]$ is semistable at least one of the $q_i(\vec{v})$ is nonzero.

1.6. **Examples.** Let us keep the following examples in mind.

(1) $V = \mathbb{C}^2$ and $G = S_2$, the group of permutations of the coordinates x, y. Then $R = \mathbb{C}[x,y]$, and $\mathbb{R}^G = \mathbb{C}[u = x + y, v = xy]$. Note that the quotient map is $(x,y) \mapsto (u = x + y, v = xy)$. Note that $\mathcal{N}$ is defined by the ideal $(x + y, xy)$ and is not reduced.

(2) Let V be the space of $n \times n$ matrices A and $G = SL(n, \mathbb{C})$ acting by conjugation. Then R is a polynomial ring, as is $\mathbb{R}^G = \mathbb{C}[u_1, u_2, \dots, u_n]$; the quotient map $A \mapsto \pi(A) = \{u_i = p_i(A)\}$, where the polynomials (in the entries of A) p_i are the coefficients of the characteristic polynomial of A:

$$det\ (\lambda_n I_n - A) = \lambda^n + p_1(A)\lambda^{n-1} + p_2(A)t^{n-2} + \dots + p_n(A)$$

(so that $p_1(A) = -tr(A)$ and $p_n(A) = (-1)^n det\ A$.) Note that $\pi(A) = \pi(B)$ iff A and B have the same characteristic polynomial iff they have the same (unordered) sets of eigenvalues. Note that p_d is homogeneous of degree d, so that scaling by $t \in \mathbb{C}$,

$$tA \mapsto (tp_1(A), \dots, t^n p_n(A))$$

Turning to the nilpotent cone, if $n = 2$, $\mathcal{N}$ is defined by the ideal $(tr(A), det\ A)$ and is a reduced affine cone of dimension 2. Note that $\mathcal{N}$ is the union of two orbits – the closed orbit consisting of the zero matrix and the dense orbit consisting of nonzero nilpotent traceless 2×2 matrices, all of which are "similar". If $n = 3$, $\mathcal{N}$ is defined by the ideal $(trace(A), trace(A^2), trace(A^3))$ and a reduced affine cone of dimension 7.

(3) Let $G = SL(2, \mathbb{C})$ act on $V = M_{4\times 2}(\mathbb{C})$ (the space of 4×2 complex matrices) by multiplying on the right. The ring of invariants is generated by the six 2×2 minors.

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \mapsto (x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_1y_4 - x_4y_1, x_2y_3 - x_3y_2, , x_2y_4 - x_4y_2, x_3y_4 - x_4y_3)$$

Note that

$$(x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3) - (x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2) + (x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2) = 0$$

so the image is a quadric cone in $\mathbb{C}^6$. The projective quotient is in fact the Grassmannian $G(2,4) \subset \mathbb{P}^5$. A matrix A corresponds to a semistable (in fact stable) point of $\mathbb{P}(M_{4\times 2}(\mathbb{C}))$ if it is of maximal rank 2, in which case the corresponding $SL(2, \mathbb{C})$ orbit can be identified with the point of $G(2,4)$ given by the image of the corresponding linear map $\mathbb{C}^4 \to \mathbb{C}^2$, $\vec{v} \mapsto A\vec{v}$. (In the earlier version of these notes and in the lecture, we had $SL(2, \mathbb{C})$ multiplying 2×4 matrices on the left.)

1.7. **Inner products and semistability.** Consider a linear action of G on V. Choose an inner product on V.

Proposition 1.2. *Let $\vec{v}$ be a nonzero vector in V. Then the corresponding point $[\vec{v}] \in \mathbb{P}(V)$ is semistable iff* $\inf_{g \in G} \|g\vec{v}\|^2 > 0$.

Proof. If $[\vec{v}] \in \mathbb{P}(V)$ is semistable, there exists a nonzero homogeneous invariant $q \in \mathbb{C}[V]$ of positive degree such that $q(\vec{v}) \neq 0$. If $\inf_{g \in G} \|g\vec{v}\|^2 = 0$, there exists a sequence $g_n \in G$ such that $\|g_n\vec{v}\|^2 \to 0$. This implies that $g_n\vec{v} \to 0_V$, and by continuity, $q(g_n\vec{v}) \to g(0_V) = 0$. But since q is invariant $q(g_n\vec{v}) = q(\vec{v}) \neq 0$, which is a contradiction.

Conversely, if if $\inf_{g \in G} \|g\vec{v}\|^2 > 0$, we know that 0_V is not contained in $\overline{G\vec{v}}^m$, the closure of the orbit $G\vec{v}$ w.r.to the topology on V given by the metric (the "classical topology"). On the other hand, by Borel's theorem, $\overline{G\vec{v}}^m$ coincides with the Zariski closure $\overline{G\vec{v}}$. Since 0_V and $\overline{G\vec{v}}$ are disjoint closed G-invariant subsets their images in $V//G$ are disjoint. (In the lecture, I gave another argument, which is essentially a proof of this part of the "good-quotient-hood" of the map $V \to V//G$.) □

Introduce a metric into complex algebraic geometry, and soon the whole camel of differential geometry is inside. This will be the theme of the last lecture. Here is a preliminary result.

Proposition 1.3. *Any critical point of* $\|.\|^2|_O$ *is a minimum. The minimum is attained on* O *iff* $\|O\|$ *is closed in* V *iff the projection of* O *to* $\mathbb{P}(V)$ *is a closed semi-stable orbit in* $\mathbb{P}(V)^{ss}$*. In this case, the minima are unique apart from the action of* $K = G \cap U(W)$*.*

The norm-squared function is defined on V (not $\mathbb{P}(V)$) and has little to say about the nilpotent cone, projectivised or otherwise.

1.8. **Unreliable history.** Everything below should be prefaced "it is my impression that.." because I have not run the fact-checks.

The first serious attention to the null-cone, was by Hilbert. His route to the finite generation of invariants was through his basis theorem (proved in this context) to describe the ideal defining the null-cone. With Mumford and his geometric invariant theory (invented as a method to construct moduli of polarised projective varieties) came the notion of GIT semistability and the discovery, most clearly in the case of moduli of vector bundles, that the notion is mirrored in an intrinsic property of the object being classified. In some sense, one "threw out" non-semistable objects, effectively ignoring the null-cone. Harder and Narasimhan, in their pioneering work, discovered the Harder-Narasimhan filtration of a non-semistable vector bundle. This intrinsic construction was linked to a stratification of the null-cone, which thus regained its relevance to moduli theory. Symplectic geometry and Morse-theoretic techniques came to the fore in the work of Atiyah-Bott, where an infinite-dimensional version of GIT was used. A celebrated theorem of Narasimhan and Seshadri was revealed as a version of the relationship (which we will see below) between minimal critical points of the norm-squared of the moment map and (poly)stable points.

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