GCT Workshop - Online lecture series
Lower bounds - Lecture 2

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Agenda:
- Nisan's lower bound for non-commutative ABPs
- Kayal's measure of shifted partial derivatives
- Some lower bounds using shifted partials and variants..

Recap: General lower bounds template:

- $\Gamma : \mathbb{F}[\bar{x}] \to \mathbb{R}$, as a proxy for size.
- For any $f \in \mathcal{C}$, show $\Gamma(f)$ is "small"
- Heuristically verify that $\Gamma(R)$ is "large" for a random R
- Find hay in a haystack.

Some common measures:

▷ $\Gamma_k(f) = \dim(\partial^{=k}(f))$

$$= \dim\left(sp \left\{ \frac{\partial^k f}{\partial m} : \begin{array}{l} m \text{ is a mon.} \\ \text{of deg } k \end{array} \right\} \right)$$

$\Gamma_k(x_1 \cdots x_n) = \binom{n}{k}$ $\qquad$ $\Gamma_k(ESym_d) \approx \min\left\{ \binom{n}{k}, \binom{n}{d-k} \right\}$

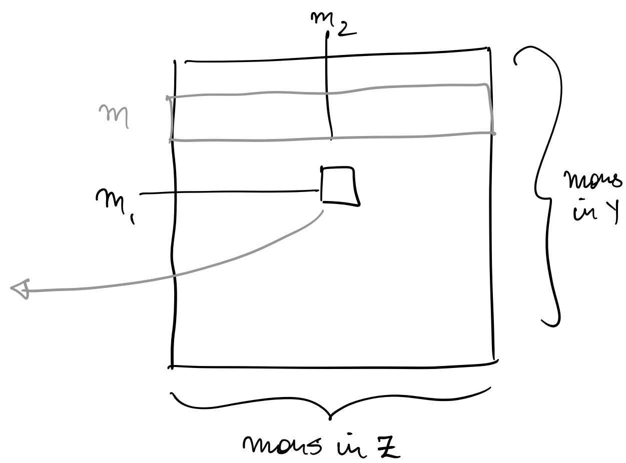
$\Gamma_k(Det_n) = \binom{n}{k}^2$

$\Gamma_k(\ell^d) = 1$ $\qquad$ $\Gamma_k(\ell_1 \cdots \ell_d) \leq \binom{d}{k}$

Useful to prove LBs for $\Sigma\wedge\Sigma$ and $\Sigma\Pi^{[d]}\Sigma$.

▷ $X = Y \sqcup Z$.

$\Gamma_{Y\sqcup Z}(f) = \mathrm{rank}\left(M_{Y\sqcup Z}(f)\right)$



$\mathrm{coeff}_f(m_1, m_2)$

$\Gamma_{Y\sqcup Z}(x_1 \cdots x_n) = 1$

$\Gamma_{Y\sqcup Z}\left(f(Y)\, g(Z)\right) = 1$ $\qquad$ $\mathrm{Row}(m) = \sigma_{Y=0}\, \partial_m(f)$

$\Gamma_{Y\sqcup Z}\left(\mathrm{ESym}_d\right) \leq d$

Very useful for multilinear & non-commutative models.

▷ Other measures:

[Baur-Strassen] $\quad \Gamma_{\bar{a}}(f) = \#\left\{\bar{x} : (\nabla f)(\bar{x}) = \bar{a}\right\}$.

$$2^s \geq \Gamma_{\bar{a}}\left(\sum x_i^d\right) \geq (d-1)^n$$

[Mignon-Ressayre] $\quad \Gamma_{x_0}(f) = \mathrm{rank}\left(\mathrm{Hess}(f_0)(x_0)\right)$

where $f(x_0) = 0$.

$\Gamma_{x_0}(f) \leq 2m$ if $dc(f) \leq m$.

$\Gamma_{x_0}(\mathrm{Perm}_n) = n^2$

Other non-natural measures:

[Kumar], [Chatterjee-Kumar-She-Volk], [Grigoriev-Karpinski]

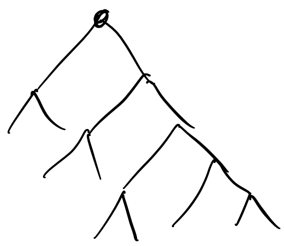
ABPs $\qquad\qquad$ etc.

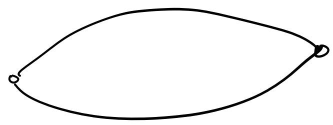
Non-commutative models:

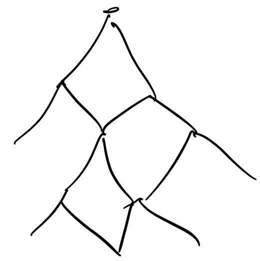
Alg. models where inputs may not commute (like say evaluated on matrices).

Stuff needs to be multiplied in the right order!

nc Formulas $\subseteq$ nc-ABP $\underset{\neq}{\subset}$ nc-Circuits



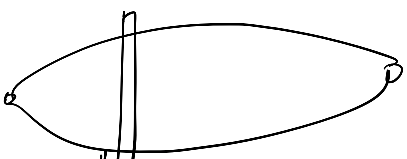




Thm: [Nisan] exp. sep between ABPs & circuits in the nc-world.

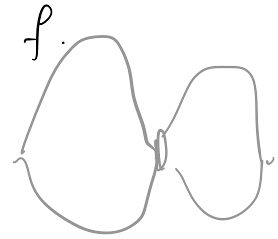
Rem: The <u>usual</u> depth redn. <u>does not</u> work.

[Nisan]: Given any hom. nc. poly f, we can <u>exactly</u> find the smallest hom. nc-ABP for f.

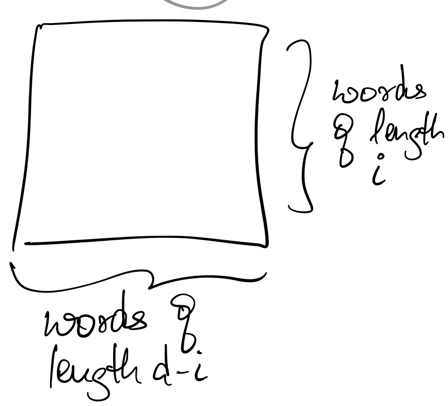


i^{th} ABP for f.

$$f = \sum_{j=1}^{w_i} g_j \cdot h_j$$



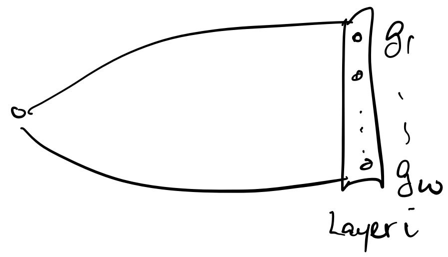
$M^{(i)}(f) =$

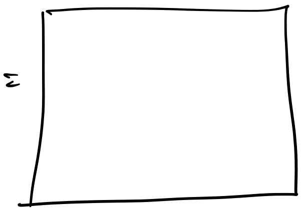


Obs: $\mathrm{rank}(M^{(i)}(f)) \leq w_i$

$$M^{(i)}(f) = \sum_{j=1}^{w_i} \boxed{g_j}\ \boxed{h_j}$$

In fact, we can build the smallest hom ABP for f.



$M^{(i)}(f) =$ 

Each colum is a poly of deg i

Inv: $g_1, .., g_w$ are columns in the matrix M_i that span all other columns.

What is column corr. to m? "Right derivative".

(Pick up all mons with "m" at its end) $\Delta_m(f) = g$.

For any m of deg $d-i$, $\Delta_m(f) = \sum \beta_j g_j$

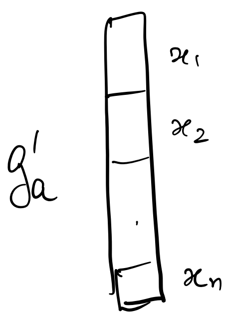
Maintaining the inv:

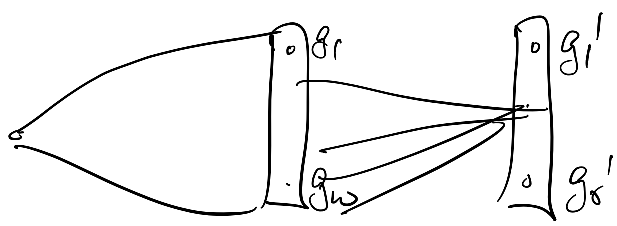
$M^{(i+1)}(f)$, and say $g_1', .., g_r'$ is a basis of columns.

$g_a' = \Delta_{m'}(f)$ where m' has deg $d-i-1$

$$= \sum_{k=1}^{n} \Delta_{x_k m'}(f) \cdot x_k$$

$$= \sum \beta_{jk} \cdot g_j \cdot x_k.$$





□.

Partial derivatives and their friends.

Recall: $2^{\Omega(n)}$ lb for $\Sigma\Lambda\Sigma$ ckts computing $x_1 \cdots x_n$

$\underset{\sum \ell_i^d}{\overset{\shortparallel}{\Sigma\Lambda\Sigma}}$

Qn: What if the model is sums of quadratics?

$$\sum Q_i^{d/2}?$$

$$\partial_x(Q^d) = Q^{d-1} \cdot \boxed{\text{lin.}}$$

$$\partial^{=k}(Q^d) = Q^{d-k} \cdot (\deg k)$$

Obs: $\underset{\Gamma_k(Q^d)}{\overset{\shortparallel}{\dim}} (\partial^{=k}(Q^d)) \leq \underline{\binom{n+k}{k}}$

$$\Gamma_k(R) \overset{\text{hope}}{\approx} \min\left\{ \underbrace{\binom{n+k}{n}}_{\text{useless.}}, \binom{n+d-k}{n} \right\}$$

In fact, $\dim \partial^{=k}\left((x_1^2 + \cdots + x_n^2)^d\right) \approx \binom{n+k}{k}$

[Kayal] What if we add shifts?

$$x^{=\ell}\, \partial^{=k}(Q^d) \subseteq sp\left\{ Q^{d-k} \cdot (\deg k+\ell) \right\}.$$

Obs: $\dim\left(x^{=\ell}\partial^{=k}(Q^d)\right) \leq \binom{n+\ell+k}{n}$

$$\Gamma_{k,\ell}(R) \overset{\text{hope}}{\approx} \min\left\{ \binom{n+k}{k}\binom{n+\ell}{n}, \binom{n+\ell+d-k}{n} \right\}.$$

... does this work?

Finding the right parameter choices is tricky...

Strangely, the "right" $\ell \geq n$...

$$\frac{\binom{n+k}{k}\binom{n+\ell}{n}}{\binom{n+k+\ell}{n}} \approx \frac{\left(\frac{n}{k}\right)^k \cdot \left(\frac{\ell}{n}\right)^n}{\left(\frac{\ell+k}{n}\right)^n} = \left(\frac{n}{k}\right)^k \cdot \left(\frac{\ell}{\ell+k}\right)^n$$

Turns out, this does work! Can give an $n^{\Omega(k)}$ lb.

Thm [Kayal] $x_1 \ldots x_n = \sum_{i=1}^{s} Q_i^{n/2}$ and $\deg(Q_i) = 2$

$\Rightarrow \quad s = 2^{\Omega(n)}$.

$k = \delta n/t$
$\ell = nt$

More generally, $x_1 \ldots x_n = \sum \underbrace{Q_i}_{\deg t}^{n/t} \Rightarrow s = 2^{\Omega(n/t)}$.

Last time, we could go from $\Sigma\Lambda\Sigma$ to $\Sigma\Pi^{[d]}\Sigma$ easily.

Does the same work here?

$$x^{=\ell}\partial^{=k}(Q_1 \ldots Q_{d/2}) \subseteq sp\left\{ \prod_{i \in S} Q_i \cdot \binom{\deg}{k+\ell} : |S| = \frac{d}{2} - k \right\}$$

Obs: $\dim x^{=\ell}\partial^{=k}(Q_1 \ldots Q_{d/2}) \leq \underbrace{\binom{d/2}{k}}_{\leq 2^{d/2} = 2^{\Theta(k)}} \cdot \binom{n+\ell+k}{n}$.

We should still get a $n^{\Omega(k)}$ lower bound. $\leftarrow$

Task: Find hay in the hay stack.

[Gupta-Kamath-Kayal-S] $2^{\Omega(\sqrt{n})}$ lb for $\Sigma\Pi^{\sqrt{n}}\Sigma\Pi^{\sqrt{n}}$ ckts computing Det_n or Perm_n.

Qn: Why don't we just engineer a poly where this calculation is easy?

dim $x^{=\ell}\,\partial^{=k}(f)$ → always monomials

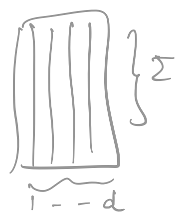
$\partial_m(f) = m_1 + m_2 +$

Can we ensure that any two mons of f have gcd $< k$.

Build a poly from a code.

$$C \subseteq \Sigma^d \longrightarrow f_C \in \mathbb{F}[x_1, .., x_m]^{=d}$$

with $m = |\Sigma| d$.



Want a code of distance $> d-k$.

$$= \sum_{(\alpha_1,..,\alpha_d)\in C} x_{1,\alpha_1} \cdots x_{d,\alpha_d}$$

Great! Now how large is $x^{=\ell}\,\partial^{=k}(f_C)$?

$\binom{d}{k}$

Fix one from each. $m_1, .., m_R$

$A_i = \{ m_i \cdot m : \; m \text{ has deg } \ell \}$.

Want to estimate $|\bigcup A_i| \geq \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j|$

[Chillara-Mukhopadhyay]. $\binom{n+\ell}{n}^{\delta}$ is small $\because$ dist.

A specific instantiation:

$$NW_{n,d,a}(x_{1,1}, \ldots, x_{1,d}) = \sum_{\substack{p \in \mathbb{F}_n[z] \\ \deg p < a}} x_{1,p(1)} \cdots x_{d,p(d)}$$

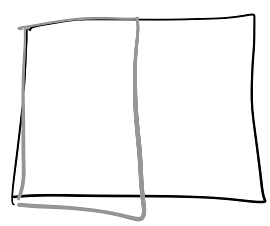
- $\in$ VNP
- has n^a monomials
- Any two monomials have gcd-deg $< a$.

Thm: [Kayal-Saha-S] Any $\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ ckt computing $NW_{n,d,k}$ requires size $n^{\Omega(\sqrt{d})}$ (for $k = \delta \cdot \sqrt{d}$)

Thm: [FLMS] Same bound for IMM.

Thm [Kumar-Saraf] Same bound for $\Sigma\Pi^{\sqrt{d}/10}\Sigma\Pi^{10\sqrt{d}}$

Variants of concern:



▷ Adding in multilinear projections.

PSPD: $\pi_{ml} \cdot x^{=\ell}\, \partial^{=k}(f)$

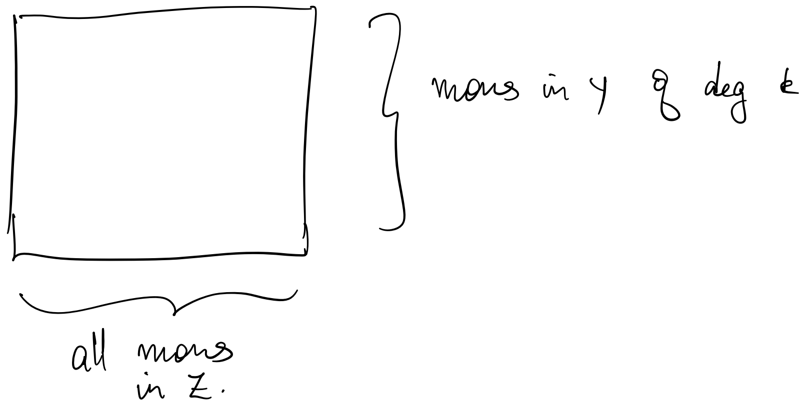
Thm: [Kayal-Limaye-Saha-Srinivasan] $n^{\Omega(\sqrt{d})}$ lb for hom. depth 4 ckts computing NW.

Thm: [Kumar-Saraf] ... for IMM (also over all fields).

▷ Adding in skew-ness : $X = Y \sqcup Z$
but diff sizes.

Skew-Partial Derivatives : [Kayal-Nair-Saha].

$|Y| >> |Z|$.



LBs for multi-k-ic models.

▷ Many other combinations of $\partial^{=k}$, $x^{=\ell}$, $\sigma_{Y=0}$, $\pi_{m.\ell}$.

[Chillara]: projected shifted skew partial derivatives.

[Garg-Kayal-Saha]: Affine projections of partials.