School on GCT, Aug – Dec 2021

Basics of algebraic geometry

Aug 27, 2021

→ Affine varieties, projective varieties, morphism.
(R. Hartshorne, Chapter I, Sections 1–3)

§1: <u>Affine varieties</u>: Work over complex numbers $\mathbb{C}$.

Geometry $\mathbb{C}^n = \{(a_1,..,a_n) \mid a_i \in \mathbb{C}\} = \mathbb{A}^n$
"Affine n-space"

Algebra $R := \mathbb{C}[X_1,..,X_n] = \{\text{poly in } n \text{ variables over } \mathbb{C}\}$

Given a polynomial $f \in R$ and $p = (a_1,..,a_n) \in \mathbb{A}^n$,
we say "<u>p is a zero of f</u>" if $f(a_1,..,a_n) = 0$

<u>eg</u>: $(1,1)$ is a zero of $x^2 - y^2$.

<u>Def</u>: Let $S \subseteq R$ be a set. The zero set of S,

$$Z(S) := \{p \in \mathbb{A}^n \mid f(p) = 0 \ \forall f \in S\} \subseteq \mathbb{A}^n$$

is an "<u>algebraic set</u>".

<u>In other words</u>: A subset $X \subseteq \mathbb{A}^n$ is an "algebraic set"

if $\exists\, S \subseteq R$ s.t $X = Z(S)$.

Rmk: If I is the ideal in R generated by S, then $Z(I) = Z(S)$. Very easy to check.

- $\left.\begin{array}{l} f, g \in R \\ p \in \mathbb{A}^n \end{array}\right\}$ $(fg)(p) = f(p)\, g(p)$

  $(f+g)\, p = f(p) + g(p)$

Fact: R is a noetherian ring, i.e; every ideal is finitely generated.

So: if $Z(S)$ is an algebraic set, then

$$Z(S) = Z(f_1, .., f_r) \text{ for } f_1, .., f_r \in R.$$

Lemma: Algebraic sets are closed under finite unions and arbitrary intersections; ϕ, $\mathbb{A}^n$ are algebraic.

Pf:
- $Z(S_1) \cup Z(S_2) = Z(S_1 S_2)$
- $\bigcap_{\alpha \in \Lambda} Z(S_\alpha) = Z\left(\bigcup_{\alpha \in \Lambda} S_\alpha\right)$
- $Z(1) = \phi$
- $Z(0) = \mathbb{A}^n$.

Conclusion: Algebraic sets are closed sets in a topology on $\mathbb{A}^n$. This is called the Zariski topology on $\mathbb{A}^n$.

Since $\mathbb{A}^n = \mathbb{C}^n$, it admits another topology, the Euclidean topology.

<u>Note</u>: Zariski closed $\Rightarrow$ Euclidean-closed.

Converse is not true!

<u>Example</u>: If $X \subseteq \mathbb{A}^1$ is Zariski closed, then X is finite or $X = \mathbb{A}^1$.

<u>Reason</u>: A nonzero poly in $\mathbb{C}[x]$ has only finitely many zeros.

But in the Euclidean top, there are many infinite closed subsets of $\mathbb{C}$ that are not equal to $\mathbb{C}$.

eg: $\{z \in \mathbb{C} \mid |z| \leq 1\}$.

<u>Def</u>: If $Y \subseteq \mathbb{A}^n$ is a subset then the ideal of Y is

$$I(Y) := \{ f \in R \mid f(p) = 0 \ \forall \, p \in Y \}$$

<u>Hilbert's Nullstellensatz</u>: Let I be an ideal in R. If $f \in R$ vanishes on all of $Z(I)$, then $f^r \in I$ for some $r \geq 1$.

So: $I(Z(I)) = \{ f \in R \mid f^r \in I \text{ for some } r \geq 1 \}$

$\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

"radical of I", $\sqrt{I}$.

<u>eg</u>: $I(Z(x_0^2)) = (x_0) = \sqrt{(x_0^2)}$

Cor: $V(I) = \phi \Rightarrow I = R$.

Reason: $V(I) = \phi \Rightarrow I(V(I)) = I(\phi) = (1)$

$$I(V(I)) = \sqrt{I}$$

$$\Rightarrow 1 \in \sqrt{I} \Rightarrow 1 \in I \Rightarrow I = R$$

This is not true for other fields: $I = (x^2+y^2+1) \in \mathbb{R}[x,y]$

$V(I) = \phi$, but $I \neq (1)$.

Prop: (1) If $X_1, X_2 \subseteq \mathbb{A}^n$ and $X_1 \subseteq X_2$, then

$$I(X_1) \supseteq I(X_2).$$

(2) If $I_1 \subseteq I_2 \subseteq R$, then $Z(I_1) \supseteq Z(I_2)$.

(3) If $I \subset R$ is an ideal, $I(Z(I)) = \sqrt{I}$.

(4) If $X \subset \mathbb{A}^n$ is any subset, $Z(I(X)) = \overline{X}$, the closure of X in the Zariski topology.

Pf: (1), (2) trivial; (3) is Hilbert's Nullstellensatz;

(4): easy.

(Def: An ideal $I \subset R$ is 'radical' if $I = \sqrt{I}$)

Upshot:

$$\left\{ \text{Algebraic sets in } \mathbb{A}^n \right\} \xleftrightarrow{\text{bijective}} \left\{ \text{radical ideals in } R \right\}$$

$$X \longmapsto I(X)$$

$$Z(I) \longleftarrow\!\!\mid I$$

This is an inclusion-reversing bijective correspondence.

Def: An affine variety is a Zariski closed subset of $\mathbb{A}^n$. (So it is simply an alg set.)

Rmk: Many authors require affine variety to be 'irreducible'.

Def: An algebraic set X is "irreducible" if it cannot be written as a union of two proper closed subsets.

Def: Let X be an affine variety. So it is a closed subset of some affine space $\mathbb{A}^n$. Then the "coordinate ring of X" is the quotient ring $\dfrac{\mathbb{C}[x_1,..,x_n]}{I(X)} =: \Gamma(X)$.

Rmk: X irreducible $\Leftrightarrow$ $I(X)$ is prime

$\Updownarrow$ easy algebraic fact.

$\Gamma(X)$ is an integral domain.

Def: A ring A is an int. domain if $a, b \in A$, $ab = 0$, then $a = 0$ or $b = 0$.

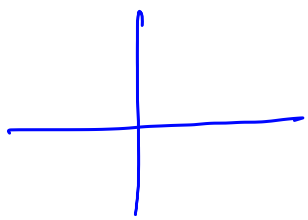
Def: An ideal $I \subseteq R$ is prime if $f, g \in R$, $fg \in I$, then $f \in I$ or $g \in I$.

eg: $(xy) \subseteq \mathbb{C}[x,y]$ is not prime.

$\because$ $xy \in (xy)$, but $x \notin (xy)$.

$y \notin (xy)$.

$$\boxed{(xy) = \{ f \cdot (xy) \mid f \in R \}}$$

$Z(xy)$:  is not irreducible.

$\quad \|$

$\{(a,b) \in \mathbb{C}^2 \mid ab = 0\}$

You can check (xy) is a radical ideal, though it is not prime.

Examples:

① $\mathbb{A}^n$ is an affine variety (it is also irreducible)

② $X = \{ (t, t^2, t^3) \mid t \in \mathbb{C} \} \subseteq \mathbb{A}^3$.

Is X algebraic?

Polynomials that vanish on X : $x_1^2 - x_2$, $x_1^3 - x_3$.

Let $I = (X_1^2 - X_2,\ X_1^3 - X_3) \subseteq \mathbb{C}[X_1, X_2, X_3]$

Then $X = Z(I)$

So X is algebraic.

$(a,b,c) \in Z(I)$

$\Rightarrow a^2 = b$

$a^3 = c$

$\Rightarrow (a,b,c) \in X$.

check: ① $I(X) = I$ because I is prime.

② $\Gamma(X) = \dfrac{\mathbb{C}[X_1, X_2, X_3]}{I} \simeq \mathbb{C}[X_1]$

③ We will see later that X is isomorphic to $\mathbb{A}^1$.

X is called a "twisted cubic curve".

③ $Y = \{ (t^2, t^3) \mid t \in \mathbb{C} \} \subset \mathbb{A}^2$ is an algebraic variety, with $I(Y) = (X_1^3 - X_2^2)$

④ Let $X = \mathbb{A}^1 - \{0\} \subseteq \mathbb{A}^1$.

Then X is an open subset of $\mathbb{A}^1$;

but X is an affine variety!

Consider $X = \mathbb{A}^1 - \{0\} \xrightarrow{\varphi} \mathbb{A}^2$

$$a \mapsto (a, a^{-1})$$

Let $Y =$ image of $\varphi = \varphi(X) \subseteq \mathbb{A}^2$.

Qu: Is Y a closed subset of $\mathbb{A}^2$?

Ans: Yes: $Y = Z(X_1X_2 - 1)$.

check: $\varphi : X \to Y$ is a bijection.

In fact, more is true: φ is a "homeomorphism".

So we can identify X with Y.

This way we can consider X as an affine variety.

Rmk: $\mathbb{A}^n - \{0\}$ does not have a structure of an affine variety for $n \geq 2$.

(5) $M_n(\mathbb{C}) = \{n \times n \text{ matrices over } \mathbb{C}\}$

$$= \mathbb{C}^{n^2}.$$

So with this identification, $M_n(\mathbb{C})$ is an affine space.

Consider $GL_n(\mathbb{C}) \subseteq M_n(\mathbb{C}) = \mathbb{C}^{n^2}$

$$\underset{\{\text{all } n\times n \text{ invertible matrices}\}}{\underset{\|}{GL_n(\mathbb{C})}}$$

We want to think of $GL_n(\mathbb{C})$ as an affine variety. But $GL_n(\mathbb{C})$ is an open subset of $M_n(\mathbb{C})$.

But
$$GL_n(\mathbb{C}) \hookrightarrow \mathbb{C}^{n^2+1}$$
$$A \longmapsto (a_{11}, \dots, a_{nn}, (\det A)^{-1})$$

Then the image is a closed subset of $\mathbb{C}^{n^2+1}$.

So it is an affine variety. Hence so is $GL_n(\mathbb{C})$.

Similarly, $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det A = 1\}$ is also an affine variety.

6 $\mathrm{Sym}^d \mathbb{C}^n := \{\text{homogeneous polynomials in } n \text{ variables of degree } d\}$

$= \mathbb{C}^{\binom{n+d-1}{d}}$

Can be identified with an affine space.

§2 : Projective varieties.

"Projective n-space" $\underline{\underline{\mathbb{P}^n}}$

Consider $\mathbb{C}^{n+1} - \{(0,0,\ldots,0)\}$ and the following equivalence relation on it :

$(a_0,\ldots,a_n) \sim (b_0,\ldots,b_n)$ if $\exists\, \lambda \in \mathbb{C}\setminus\{0\}$ s.t.

$b_i = \lambda a_i \;\; \forall\, i = 0,\ldots,n.$

$\mathbb{P}^n$ = equivalence classes = $\mathbb{C}^{n+1} - \{0\} \big/ \sim$

$\parallel$

set of lines through the origin in $\mathbb{C}^{n+1}$.

$\therefore$ A point $p \in \mathbb{P}^n$ is a line in $\mathbb{C}^{n+1}$ through the origin.

It can be represented by $[a_0 : a_1 : \ldots : a_n]$, $a_i \in \mathbb{C}$.

eg: $\mathbb{P}^2 \ni [1:2:3] = [i:2i:3i]$

We want to view points of $\mathbb{P}^n$ as zero of polynomials in $\mathbb{C}[X_0,\ldots,X_n]$.

$f = X_0^2 + X_1$; $[1:2] \in \mathbb{P}^1$: $f([1:2]) = 1+2 = 3$

$$\begin{array}{ccc} f([1:2]) & = & 3 \\ \parallel & & \neq \\ f([2:4]) & = & 8 \end{array}$$