Affine Varieties, projective varieties, morphism.
(R. Hartshorne, Chapter I, Sections 1–3)

§1: Affine Varieties: Work over complex numbers $C$.

Geometry: $C^n = \{ (a_1, \ldots, a_n) \mid a_i \in C \} = \mathbb{A}^n$

"Affine $n$-space"

Algebra: $R : = C[X_1, \ldots, X_n] = \{ \text{poly in } n \text{ variables over } C \}$

Given a polynomial $f \in R$ and $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$,
we say "$p$ is a zero of $f$" if $f(a_1, \ldots, a_n) = 0$

eg.: $(1, 1)$ is a zero of $x^2 - y^2$.

Def.: Let $S \subseteq R$ be a set. The zero set of $S$,
$Z(S) : = \{ p \in \mathbb{A}^n \mid f(p) = 0 \text{ for all } f \in S \} \subseteq \mathbb{A}^n$

is an "algebraic set".

In other words: A subset $X \subseteq \mathbb{A}^n$ is an "algebraic set"
if \( \exists S \subseteq R \text{ s.t. } X = Z(S) \).

\[ \text{Rmk: If } I \text{ is the ideal in } R \text{ generated by } S, \text{ then } \]
\[ Z(I) = Z(S). \text{ Very easy to check.} \]

\[ \cdot f, g \in R \implies (fg)(p) = f(p)g(p) \]
\[ \forall p \in \mathbb{A}^n \implies (f+g)(p) = f(p) + g(p) \]

\[ \text{Fact: } R \text{ is a noetherian ring, i.e., every ideal is finitely generated.} \]

So: \( \text{if } Z(S) \text{ is an algebraic set, then } \]
\[ Z(S) = Z(f, ..., f_r) \text{ for } f, ..., f_r \in R. \]

\[ \text{Lemma: Algebraic sets are closed under finite unions and arbitrary intersections; } \phi, \mathbb{A}^n \text{ are algebraic.} \]

\[ \text{Pf: } Z(S_1) \cup Z(S_2) = Z(S_1S_2) \]
\[ \cap Z(S_a) = Z(U S_a) \]
\[ Z(1) = \phi \]
\[ Z(0) = \mathbb{A}^n. \]

Conclusion: Algebraic sets are closed sets in a topology on \( \mathbb{A}^n \). This is called the Zariski topology on \( \mathbb{A}^n \).
Since $\mathbb{A}^n = \mathbb{C}^n$, it admits another topology, the Euclidean topology.

Note: Zariski closed $\Rightarrow$ Euclidean-closed. Converse is not true!

Example: If $X \subseteq \mathbb{A}^1$ is Zariski closed, then $X$ is finite or $X = \mathbb{A}^1$.

Reason: A non-zero poly in $\mathbb{C}[x]$ has only finitely many zeros. But in the Euclidean top., there are many infinite closed subsets of $\mathbb{C}$ that are not equal to $\mathbb{C}$.

eg: $\{z \in \mathbb{C} | |z| < 1\}$.

Def: If $Y \subseteq \mathbb{A}^n$ is a subset then the ideal of $Y$ is

$I(Y): = \{f \in R | f(p) = 0 \text{ for all } p \in Y\}$

Hilbert's Nullstellensatz: Let $I$ be an ideal in $R$.

If $f \in R$ vanishes on all of $Z(I)$, then $f^r \in I$ for some $r \geq 1$.

So: $I(Z(I)) = \{f \in R | f^r \in I \text{ for some } r \geq 1\}$

"radical of $I" = \sqrt{I}$.

eg: $I(Z(x_0^2)) = (x_0) = \sqrt{(x_0^2)}$
**Cor:** \( \text{V}(I) = \emptyset \Rightarrow I = R \).

**Reason:** \( \text{V}(I) = \emptyset \Rightarrow \text{IC}(\text{V}(I)) = \text{IC}(\emptyset) = (1)^{-1} \)
\( \Rightarrow 1 \in \sqrt{I} \Rightarrow 1 \in I \Rightarrow I = R \)

This is not true for other fields: \( I = (x^2 + y^2 + 1) \in \mathbb{R}[x, y], \text{V}(I) = \emptyset \), but \( I \neq (1) \).

**Prop:** (1) If \( X_1, X_2 \subseteq \mathbb{A}^n \) and \( X_1 \subseteq X_2 \), then \( I(X_1) \subseteq I(X_2) \).

(2) If \( I_1 \subseteq I_2 \subseteq R \), then \( Z(I_1) \supseteq Z(I_2) \).

(3) If \( I \subseteq R \) is an ideal, \( I(Z(I)) = \sqrt{I} \).

(4) If \( X \subseteq \mathbb{A}^n \) is any subset, \( Z(I(Z(X))) = \overline{X} \), the closure of \( X \) in the Zariski topology.

**Pf:** (1), (2) trivial; (3) is Hilbert's Nullstellensatz;
(4): easy.

(Def: An 'ideal \( IR \) is 'radical' if \( I = \sqrt{I} \))

**Upshot:**
\[
\begin{align*}
\{ \text{Algebraic Sets in } \mathbb{A}^n \} & \overset{\text{bijective}}{\leftrightarrow} \{ \text{radical ideals in } R \} \\
X & \rightarrow I(X) \\
Z(I) & \leftarrow I
\end{align*}
\]

This is an inclusion-reversing bijective correspondence.
Def: An **affine variety** is a Zariski closed subset of $\mathbb{A}^n$. (So it is simply an alg set.)

Rmk: Many authors require affine variety to be `irreducible'.

Def: An algebraic set $X$ is "irreducible" if it cannot be written as a union of two proper closed subsets.

Def: Let $X$ be an affine variety. So it is a closed subset of some affine space $\mathbb{A}^n$. Then the "coordinate ring of $X$" is the quotient ring $\frac{\mathbb{C}[X_1, \ldots, X_n]}{I(X)} =: \Gamma(X)$.

Rmk: $X$ irreducible $\iff I(X)$ is prime

\[ \uparrow \] easy algebraic fact.

\[ \Gamma(X) \text{ is an integral domain.} \]

Def: A ring $A$ is an **int. domain** if $a,b \in A, \ ab = 0$, then $a = 0$ or $b = 0$. 

Def: An ideal \( I \subseteq R \) is prime if for \( f, g \in R \), \( fg \in I \), then \( f \in I \) or \( g \in I \).

Eg: \( (xy) \subseteq \mathbb{C}[x,y] \) is not prime.

\[ (xy) = \{ f \cdot (xy) \mid f \in R \} \]

\( \mathbb{Z}(xy) \) is not irreducible.

\[ \{ (a,b) \in \mathbb{C}^2 \mid ab = 0 \} \]

You can check \((xy)\) is a radical ideal, though it is not prime.

Examples:

1. \( \mathbb{A}^n \) is an affine variety (it is also irreducible).
2. \( X = \{ (t, t^2, t^3) \mid t \in \mathbb{C} \} \subseteq \mathbb{A}^3 \).

Is \( X \) algebraic?

Polynomials that vanish on \( X \):
\[ x_1^2 - x_2, x_1^3 x_3. \]
Let \( I = (x_1^2 - x_2, x_1^3 - x_2) \subseteq \mathbb{C}[x_1, x_2, x_3] \)

Then \( X = Z(I) \)

So \( X \) is algebraic.

Check: \( \text{I}(X) = I \) because \( I \) is prime.

\( 1 \) \( \Gamma(X) = \frac{\mathbb{C}[x_1, x_2, x_3]}{I} \subseteq \mathbb{C}[x_1] \)

\( 2 \) We will see later that \( X \) is isomorphic to \( \mathbb{A}^1 \).

\( X \) is called a "twisted cubic curve."

\( 3 \) \( Y = \{ (t^2, t^3) \mid t \in \mathbb{C} \} \subseteq \mathbb{A}^2 \) is an algebraic variety, with \( \text{I}(Y) = (x_1^3 - x_2^2) \)

\( 4 \) Let \( X = \mathbb{A}^1 - \{(0)\} \subseteq \mathbb{A}^1 \).

Then \( X \) is an open subset of \( \mathbb{A}^1 \).

But \( X \) is an affine variety!
Consider $X = \mathbb{A}^1 - \{0\} \xrightarrow{\varphi} \mathbb{A}^2$

$a \mapsto (a, a^{-1})$

Let $Y = \text{image of } \varphi = \varphi(X) \subseteq \mathbb{A}^2$.

Qn: Is $Y$ a closed subset of $\mathbb{A}^2$?

Ans: Yes: $Y = \mathbb{Z} (x_1 x_2 - 1)$. 

Check: $\varphi: X \to Y$ is a bijection.

In fact, more is true: $\varphi$ is a "homeomorphism".

So we can identify $X$ with $Y$.

This way we can consider $X$ as an affine variety.

Rmk: $\mathbb{A}^1 - \{0\}$ does not have a structure of an affine variety for $n \geq 2$.

5) $M_n(\mathbb{C}) = \{n \times n \text{ matrices over } \mathbb{C}\}$

= $\mathbb{C}^{n^2}$

So with this identification, $M_n(\mathbb{C})$ is an affine space.
Consider \( G_{\text{Ln}}(\mathbb{C}) \subseteq M_n(\mathbb{C}) = \mathbb{C}^{n^2} \)
\( \{ \) all \( n \times n \) invertible matrices \( \} \)

We want to think of \( G_{\text{Ln}}(\mathbb{C}) \) as an affine variety. But \( G_{\text{Ln}}(\mathbb{C}) \) is an open subset of \( M_n(\mathbb{C}) \).

But \( G_{\text{Ln}}(\mathbb{C}) \hookrightarrow \mathbb{C}^{n^2+1} \)

\[ A \mapsto (a_{11}, \ldots, a_{nn}, (\text{det} A)^{-1}) \]

Then the image is a closed subset of \( \mathbb{C}^{n^2+1} \).
So it is an affine variety. Hence so is \( G_{\text{Ln}}(\mathbb{C}) \).

Similarly, \( \text{SL}_n(\mathbb{C}) = \{ A \in M_n(\mathbb{C}) \mid \text{det} A = 1 \} \)
is also an affine variety.

\( \circ \) \( \text{Sym}^d \mathbb{C}^n := \{ \) homogeneous polynomials in \( n \)
\( \text{variables of degree } d \} \)

\( \left( \begin{array}{c}
\text{can be} \\
\text{identified with an} \\
\text{affine space}
\end{array} \right) \)
§2: Projective varieties

"Projective n-space" $\mathbb{P}^n$

Consider $\mathbb{C}^{n+1} \setminus \{(0,0,\ldots,0)\}$ and the following equivalence relation on it:

$$(c_0,\ldots,c_n) \sim (b_0,\ldots,b_n) \text{ if } \exists \lambda \in \mathbb{C} \setminus \{0\} \text{ s.t. }$$

$$b_i = \lambda c_i \text{ for } i = 0,\ldots,n.$$ 

$\mathbb{P}^n = \text{ equivalence classes } = \mathbb{C}^{n+1} \setminus \{(0,0,\ldots,0)\}$

is the set of lines through the origin in $\mathbb{C}^{n+1}$.

A point $p \in \mathbb{P}^n$ is a line in $\mathbb{C}^{n+1}$ through the origin.

It can be represented by $[c_0:a_1:\ldots:a_n]$, $a_i \in \mathbb{C}$.

$\mathbb{P}^2 \ni [1:2:3] = [a:2a:3a]$ if $a \in \mathbb{C}$.

We want to view points of $\mathbb{P}^n$ as zeros of polynomials in $\mathbb{C}[x_0,\ldots,x_n]$.

$$f = x_0^2 + x_1 \quad [1:2] \in \mathbb{P}^1 \quad f([1:2]) = 1+2 = 3$$

$$f([2:4]) = 8$$