Algebraic algorithms for null cone membership for left-right action.

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EDMOND's:

Symbolic rank

\[
T \rightarrow \alpha_1 x_1 + \cdots + \alpha_n x_n
\]

Decision version SDIT.
EDMOND'S in NON COMMUTATIVE SETTING.

\[ \mathcal{F} \langle x \rangle - \text{free skewfield} \]

\( \text{ncrank} \) - non commutative rank.

Cohn's definition:

\[
T = \begin{pmatrix}
\vdots \\
m \\
\vdots 
\end{pmatrix}
\begin{pmatrix}
\vdots \\
s \\
\vdots 
\end{pmatrix}
\begin{pmatrix}
\vdots \\
m \\
\vdots 
\end{pmatrix}
\]

\[ \text{rank}(T) \leq \text{ncrank}(T) \]
Equivalent Formulation:

Cohn, Fortan-Reutenauer

Gurvits, Hrubes & Wigderson

\( M(n, F) \)

**Matrix Spaces**

Linear subspace of \( M(n, F) \)

\[ T = x_1 B_1 + \ldots + x_m B_m , \quad B_i \in \text{span} (B) \]

\[ \text{If} \quad |B| > n \quad \text{then} \quad \text{rank} (T) = \text{rank} (B) \]

\[ \text{rank} (T) \text{ also corresponds to a property of } B \]
Def: c-shrunk subspace; \( \mathcal{B} = \langle B_1, \ldots, B_m \rangle \); we say \( U \subset \mathbb{F}^n \) is \( c \)-shrunk if \( \forall W \subset \mathbb{F}^n, \dim W \leq \dim U \) and \( \forall B \in \mathcal{B}, \quad B(U) \leq W \)

2. Largest \( c \) for which \( \mathcal{F} \) is a \( c \)-shrunk subspace.

When \( \mathcal{F} = c \);

- rank decreasing completely positive operator
- \( c = \text{cone of positive } c \)-matrices (psd)
- \( P: \mathcal{C} \to P(c) = \sum B_i C B_i^+ \)
Null cone membership under \(SL(n) \times SL(n)\) action,

\[
\begin{pmatrix}
B_1 & \\
B_2 & \\
\vdots & \\
B_m & 
\end{pmatrix}
\xrightarrow{\oplus m}
\begin{pmatrix}
xb_1 y & \\
xb_2 y^t & \\
\vdots & \\
xb_m y^t
\end{pmatrix}
\]

Action on \(Mat_n\)

4. \[
\frac{\text{rank}(B_{[a]}^c)}{d} \geq \frac{\text{rank}(B_{[a]}^c)}{d-1} \geq \ldots
\]

\[
B \cdot B_{[a]} \preceq \langle B_1 \otimes \eta(\beta_1^F), B_2 \otimes \eta(\beta_2^F), \ldots \rangle
\]
Exp. Regularly lemma;

DM Poly time algorithm; get bound on max degree of any y in V, O(n)

IGS Polytime;

GOW Check for orbit closure;

DM
Motivation from GIT:

\[ G = \text{SL}(n) \times \text{SL}(n); \]

\[ M_m = \text{m-tuples of } n \times n \text{ matrices;} \]

\[ G \triangleright M_m \]

Ring of invariants:

\[ f : \text{Mat}^m \rightarrow \mathbb{C}; \]

\[ f(g \cdot b) = f(b) \]
Theorem: Let $X_i - \text{dual of } M_i$, $i = 1, \ldots, m$.

Let $A_1, \ldots, A_m \in M(d, \mathbb{F})$.

$$\det \left( A_1 \otimes x_1 + A_2 \otimes x_2 + \ldots + A_m \otimes x_m \right)$$

is invariant, and all invariants are generated by such invariants;

$\text{DW, S\&VdB, DZ, ANS, } \ldots$.

Question: Upper bound on $d$ needed for generation? 2
• GGOW - poly(1/3') ✓
• AZGLOW
• BF40OW ✓

Edmonds problem:
• Gurvits (Edmonds-lab)
• IKQS - Gurvits way

Q: lack 1 channel matrix, Faallo?
\[ B_1, \ldots, B_m \]

Suppose \( \exists X, Y \) s.t.

\[ A B, Y = \]

\[ A B', Y = \]

Block \( \Delta \), \( k \times l \) zero block.

\[ k + l \geq n. \]
- $\text{ncrank} A(x) = n - \max \{ \dim v' - \dim U, U_i, A_i U \leq U' \}$

**Hall Blocker:**

\[
\text{rank}(B) \leq \text{ncrk}(B)
\]

- **Ex:**

$2 \times 3$

Skew symmetric matrix of odd degree; sk uk ncrank

\[
\begin{pmatrix}
1 & 1 \\
2 & 3
\end{pmatrix}
\]
• Second Wong sequence:

Notation: \( B = \langle B_1, \ldots, B_m \rangle \)

\( \text{rank } (B) : \)

\( \text{corank } (B) : \)

• For \( U \subseteq \mathbb{F}^n \), \( B(U) = \{ B(u) \mid B \in B, u \in U \} \)

\( U \) is a \( c \)-singularity witness if \( \lim_{n \to \infty} \dim (B(u)) \subset \dim U - c \)

\( \text{disc } (B) = n - c, \quad c \) is the largest \( \in \) which \( U \) is a \( c \)-shrink subspace

Clearly \( \text{corank } (B) \geq \text{disc } (B) \)
Let $A \in \langle B_1, \ldots, B_m \rangle$.

Let $v$ s.t. $Av = 0$, $v \in \ker(A)$.

SECOND WONG SEQUENCE of $(A, B)$
\[ W_{i+1} = B(\bar{A}(\omega_i)) \]

**Check:**
- \( W_{i+1} = \omega_i \) if \( i \in I \);
- \( W_{i+1} = \omega_i \) if \( B(\omega_i) = \bar{A}(\omega_i) \);
- Stabilizes:

**Lemma:** Let \( A \in \mathbb{B} \). Let \( \omega^* \) be the limit of the second Wong sequence \( q(\bar{A}, \omega) \). Then \( \omega^* \) is a corank \( (A) \) singularity witness of \( \mathbb{B} \) if \( \omega^* \in \text{Im}(A) \).

In this case, \( A \) is of maximal rank in \( \mathbb{B} \) and \( \bar{A}(\omega^*) \) is a corank \( (\mathbb{B}) \) singularity witness.
**Proof:**

If \( w^* \subseteq \text{Im}(A) \), then

\[
\dim(A^{-1}(w^*)) = \dim(w^*) + \dim(\ker(A))
\]

But \( w^* = B(A^{-1}(w^*)) \)

\[
\dim(w^*) = \dim(\ker(A))
\]

\[
A^{-1}(w^*) \text{ is a } \dim(\ker(A))-\text{singularly witness.}
\]
Suggest an augmenting path algorithm:

\[
\text{If } \nu^* \subseteq \text{im}(A) \quad - \quad \text{done}
\]

O-w.
\[ B_{i2} = A_1 B_{i2} A_2 B_{i1} \kappa \epsilon A \notin \text{Im}(A) \]

Simplify: Assume

\[ A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix} \]

\[ \therefore B_{i2} B_{i1} \ldots B_{i1}(v) \notin \text{Im}(A) \]
If \( l = 1 \), then

\[
\det (x A + B) = 0
\]

\[
\det \begin{pmatrix}
x \\
x \\
x \\
x \\
\end{pmatrix} = b x + \ldots
\]

Try:

\((\sim)\)

All possible values of \( F \),

Increase the scale.
If $B_{i_1} = B_{i_2} = B$ then

$$B = \begin{pmatrix} * & * \\ * & b \end{pmatrix}$$

Can choose a Basis $B_{v_1}, B_{v_2}, \ldots, B_{v_j}$

$BA \in \text{Im}(A)$

$xA + B$ looks like

$$\begin{pmatrix} x & \ast & \ast & \ast & \ast & \ast & \ast & 1 \\ x & \ast & \ast & \ast & \ast & \ast & \ast & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ldots & \ast & \ast & x & y & \ast & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
Write $y = x + c$, $z = x + d$, $-$ $-$

$$\det(xA + B) = x^{n-l} + \text{lower 2nd order terms}.$$ 

Can you reduce to this case $^2$?

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$$B = \left(x_1 B_1 + \cdots + x_l B_l \right),$$

$$B^* (n,n) = b_1 x_1 + \cdots + x_l \text{ other terms.}$$

$^{\text{degree } l}$

PROBLEM: Interference
$$\text{Example:}$$

\[A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\]

\[\text{by}\]

\[\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 \end{pmatrix}\]

\[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}\]

\[\Rightarrow \]

Start with \(A_1\):

\[A_2^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\]

\[A_2^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\]
\( A_2A_3 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \)

\( A_2A_2 = \begin{pmatrix} 0 & 0 & \circ \\ \circ & -1 & \circ \\ \circ & \circ & 0 \end{pmatrix} \)

\( (xA_2 + yA_3)^2 \)  unfortunately

\[
\begin{pmatrix}
\end{pmatrix}
\]

But...

Skew symmetric
Work with $x, y$ non-commuting;

We may not not be able to reach in $M(n, jT)$ but maybe in an ext;

In this example

$H = \mathbb{R} \langle x, y, z \rangle / (x^2 + 1, y^1, xy - 1, yx + x)$

Hence:

$(xA_2 + yA_3)^2 = \begin{pmatrix} z & -1 & \varepsilon \\ 1 & \varepsilon & \varepsilon \\ 0 & -x & 0 \end{pmatrix}$

and:

$A_1 + xA_2 + yA_3 = \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ -y & -x & 0 \end{pmatrix}$
HAS FULL RANK.

\[ \psi : H \rightarrow M_2(\mathbb{C}) \]

\[ x \rightarrow \begin{pmatrix} i & -1 \\ -i & 1 \end{pmatrix} \]

\[ y \rightarrow \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \]

\[ z \rightarrow \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \]

Then

\[ (A_1 + xA_2 + yA_3) \rightarrow \begin{pmatrix} 1 & -x \\ -y & 1 \end{pmatrix} \]

\[ \begin{pmatrix} 1 & -x \\ -y & 1 \end{pmatrix} \text{ has full rank.} \]
Rank has increased not in $\mathbb{B}$ but $B^{[2]}$ - second blow up.

• A general phenomenon;

• $\text{Bie } \text{Bie }$ has an entry in $(n,n)$

• Choose $d > l$

\[ A = A \otimes \text{Id} \]

\[ B = B_{1} \otimes E_{21} + B_{2} \otimes E_{32} + \ldots + B_{n} \otimes E_{n+l+1,l} \]

$\nu \in \ker(A)$.
\[ v \otimes u_1 \in \ker(A') \]

\[ e \triangleleft (v, \otimes u_1) \text{ has a term} \]

\[ B_i \otimes E \delta_{l+1,2} \]

\[ = B_{i-1} (v) \otimes E_{l+1,1} (u_1) \]

- The second Wong sequence \( \langle A', B \rangle \)
- \( \omega \not\in \text{Im}(A') \)

\[ \therefore \quad \lambda A' + B \text{ has rank} \]

\[ \geq 2d+1 \]

\[ \geq rd+1 \]

\[ \geq rd+d \]

\[ d(\gamma+1) \]
Regularity lemma y boils up:

If $A$ is a linear subspace of matrices $\sigma_k(A^d)$ is a multiple of $d$

\[ \text{Pf: (DM)}: \]

\[ A = X_0 + t_i X_1 + \ldots + t_m X_m \]

\[ \mathcal{O} = \langle X_1, \ldots, X_m \rangle \]

\[ \sigma_k(\mathcal{O}) = \sigma_k(X_0 \otimes I + X_1 \otimes T_i + \ldots + X_m \otimes T_m) \]

$T_i$'s generic

\[ R_d \subseteq \text{Mat}_{d,d}(K[[t_{i,k}^{\pm}]] \)
Zl - center of Re
Ql - field fraction of Zl

Amplitude: Ql ⊗ Rl is a div alg
Universal div alg of degree d

X₀ ⊗ I + X₁ ⊗ T₁ + X₂ ⊗ T₂ + ... + Xₘ ⊗ Tₘ
is of from

\[
\begin{bmatrix}
    d & d & & \\
    & d & & \\
    & & \ddots & \\
    & & & d
\end{bmatrix}
\]

Now do some of operations in Br ∂ Br
Can round up rank from \( \delta d + \Delta \)

\[
\delta d + 1 = (\delta + 1)d;
\]

Can do this algorithmically in poly time by working in central division algebras.

[**Thm.**] Let \( B \leq M(n, F) \); \( A = \mathcal{O}B \)

Assume \( A \in A \), \( \sigma k(A) = \gamma d \).

Let \( d' \geq \delta \), for large \( |F| \), \( \exists \) a det algorithm that returns a \((n-\delta)d\) shoreline subspace for \( A \).

or \( A^* \in A \oplus M(d', F) \) of rank \((\delta + 1)d')d'\)
DM show null cone is out by poly of degree $\leq n^2 - n$

**Obs:** $\mathbb{B}^d$ contains a nonsingular matrix for $d \leq n+1$.

**Pf:** Assume $d > n+1$

\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{e1} & A_{e2} & A_{e3} & A_{e4}
\end{bmatrix}
\]

\[\text{nonsingular}\]
$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$

Nonsingular
\[ dn - 2n = dn - \omega + 1 - (n+1) \]

\[ > dn - n + 1 - d \]

\[ = d(n-1) - (n-1) \]

\[ = (d-1)(n-1) \]

Blue \in \mathcal{B}^{[d-1]} \]

\[ \therefore \gamma_k (\text{Blue}) = (d-1)n \]

\[ \text{Full rank} \]