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Algebraic algorithms for null cone membership for left-right action.

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(joint with Youming Qiao, Gabor Ivanyos)

EDMOND's:

Symbolic rank $\qquad X = \{x_1 \dots x_m\}$

$$\begin{array}{cc} & m \\ n & \begin{bmatrix} T & \bullet \end{bmatrix} \end{array} \rightarrow a_1 x_1 + \cdots + a_n x_m$$

Decision version **SDIT**.

EDMOND'S in NON COMMUTATIVE SETTING.

$\mathbb{F}\langle x \rangle$ - free skew field

ncrk - non commutative rank.

<u>Cohn's definition:</u>

$$T = \underset{m}{\left(\qquad \right)}^{s} \underset{\underline{s}}{\left(\qquad \right)}^{m}$$

$$\text{rank}(T) \leq \text{ncrank}(T)$$

EQUIVALENT FORMULATION:

Cohn, Fortin-Reutenauer

$\underbrace{\text{Gurvits,}}$ Hrubes & Wigderson

$M(n, \mathbb{F})$

MATRIX SPACE:

Linear subspace of $M(n, \mathbb{F})$

$T = x_1 B_1 + \cdots + x_m B_m, \quad B_i \in M(n, \mathbb{F})$

$\underbrace{\mathcal{B} = \langle B_1, \dots, B_m \rangle}$

$|\mathbb{F}| > n$ then $\operatorname{rk}(T) = \underbrace{\operatorname{rk}(\mathcal{B})}$

ncrank(T) also corresponds to a property of $\mathcal{B}$

1

Def: c-Shrunk subspace;

$\mathcal{B} = \langle B_1, \dots, B_m \rangle$; We say $U \subseteq \mathbb{F}^n$ is c-shrunk if $\exists\, W \subseteq \mathbb{F}^n$, $\dim W \le \dim U - c$ and $\forall\, B \in \mathcal{B}$, $B(U) \le W$

Q: Largest c for which $\exists$ a c-shrunk subspace;

2

When $\mathbb{F} = \mathbb{C}$;

rank decreasing completely positive operators

$\mathcal{C}$ = cone of positive s-matrices (psd)

$P : \mathcal{C} \longrightarrow$ $P(C) = \sum B_i C B_i^{\dagger}$

Ψ

C

3. Null cone membership under $SL(n) \times SL(n)$ action,

$(X, Y) \in SL(n) \times SL(n)$

$$\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix} \longrightarrow \begin{pmatrix} XB_1Y^T \\ XB_2Y^T \\ \vdots \\ XB_mY^T \end{pmatrix}$$

action on $\mathrm{Mat}_n^{\oplus m}$

4. $$\underbrace{\frac{\mathrm{rank}(\mathcal{B}^{[d]})}{d}} \geq \frac{\mathrm{rank}(\mathcal{B}^{[d-1]})}{d-1} \geq \cdots$$

$\mathcal{B}$: $\mathcal{B}^{[d]} \triangleq \langle B_1 \otimes M(d, \mathbb{F}), B_2 \otimes M(d, \mathbb{F}), \ldots \rangle$

IQS

Exp. Regularity lemma;

GGOW.

✓ C;

DM Poly time algorithm; Get bounds on max degree of ring of inv; $O(n^6)$

IQS Polytime;

GGOW Check for orbit closure;

DM

- Motivation from GIT:

$G = SL(n) \times SL(n)$;

$M_m =$ m-tuples of $n \times n$ matrices;

$G \curvearrowright M_m$

- Ring of invariants:

$$\left.\begin{aligned} f : \mathrm{Mat}^{\oplus m} &\longrightarrow \mathbb{C}, \\ f(g \cdot \underline{b}) &= f(b) \end{aligned}\right\},$$

THM: Let X_i - dual of M_i, $i = 1, \dots m$,

let $A_1, \dots A_m \in M(d, \mathbb{F})$;

$\det(A_1 \otimes X_1 + A_2 \otimes X_2 + \dots + A_m \otimes X_m)$

is invariant, and all invariants are

generated by such invariants;

DW, S & VdB, DZ, ANS, ...

Q: upper bound on d needed for

generation?

- GGOW – poly($1/\varepsilon$) ✓
- AZGLOW
- BFGOWW ... ✓

<u>Edmonds problem:</u>

- Gurvits (Edmonds–Rado)

Q: Rank 1 spanned matrix spaces?

- IKQS – Generalized Wong seq

- $\underline{B_1, \dots, B_n}$.

Suppose $\exists\, \underline{X, Y}$ s.t.

$$AB_1Y = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$$

$$AB_2Y = \begin{array}{cc} & \quad\;\; l \\ k & \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \end{array}$$

$$\vdots$$

Block Δ, $k \times l$ zero block;

$$\underline{k + l \geq n}.$$

- $\operatorname{ncrank} A(x) := n - \max\{\dim V' - \dim U,$
$\forall i,\ A_i U \leq U'\}$

HALL BLOCKER:

$\operatorname{rank}(\mathcal{B}) \leq \operatorname{ncrk}(\mathcal{B})$

- Ex:

3×3

Skew symm matrices of std type;

rk vs ncrank

$$\operatorname{rk} = 2 \qquad \operatorname{ncrank} = 3$$

• Second Wong sequence:

Notation: $\mathcal{B} = \langle B_1, \ldots, B_m \rangle$

rank$(\mathcal{B})$:

• corank$(\mathcal{B})$:

• For $U \subseteq \mathbb{F}^n$, $\mathcal{B}(U) = \langle B(u) \mid B \in \mathcal{B}, u \in U \rangle$

U is a c-singularity witness if $\dim \langle \mathrm{im}(B(U)) \rangle \leq \dim U - c$;

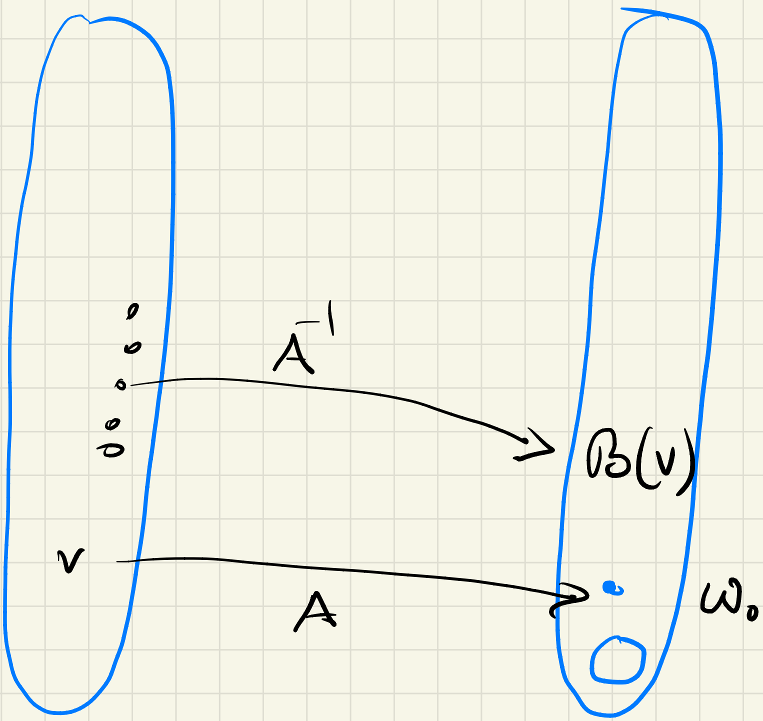
• disc$(\mathcal{B}) = n - c$, c is the largest for which $\exists$ a c-shrunk subspace;

Clearly corank$(\mathcal{B}) \geq$ disc$(\mathcal{B})$;

Let $A \in \langle B_1, \dots, B_m \rangle$.

Let v s.t. $Av = 0$, $v \in \ker(A)$

SECOND WONG SEQUENCE of $(A, \mathcal{B})$



$$W_{i+1} = \mathcal{B}\left(A^{-1}(W_i)\right)$$

CHECK:

- $W_{i+1} \supseteq W_i \quad \forall i$;
- $W_{i+1} = W_i$ iff $\mathcal{B}^{-1}(W_i) \supseteq A^{-1}(W_i)$
- Stabilizes:

LEMMA: Let $A \in \mathcal{B}$, let W^* be the limit of the second Wong sequence of $(A, \mathcal{B})$. Then $\exists$ a corank(A) singularity witness of $\mathcal{B}$ iff $W^* \subseteq \mathrm{Im}(A)$; In this case A is of maximal rank in $\mathcal{B}$ and $A^{-1}(W^*)$ is a corank$(\mathcal{B})$ singularity witness.

Proof:

If $W^* \subseteq \mathrm{Im}(A)$

$$\dim\left(A^{-1}(W^*)\right) = \dim(W^*) + \dim(\ker(A))$$

But $W^* = B\left(A^{-1}(W^*)\right)$

$$\dim W^* + \dim\left(\ker(A)\right)$$

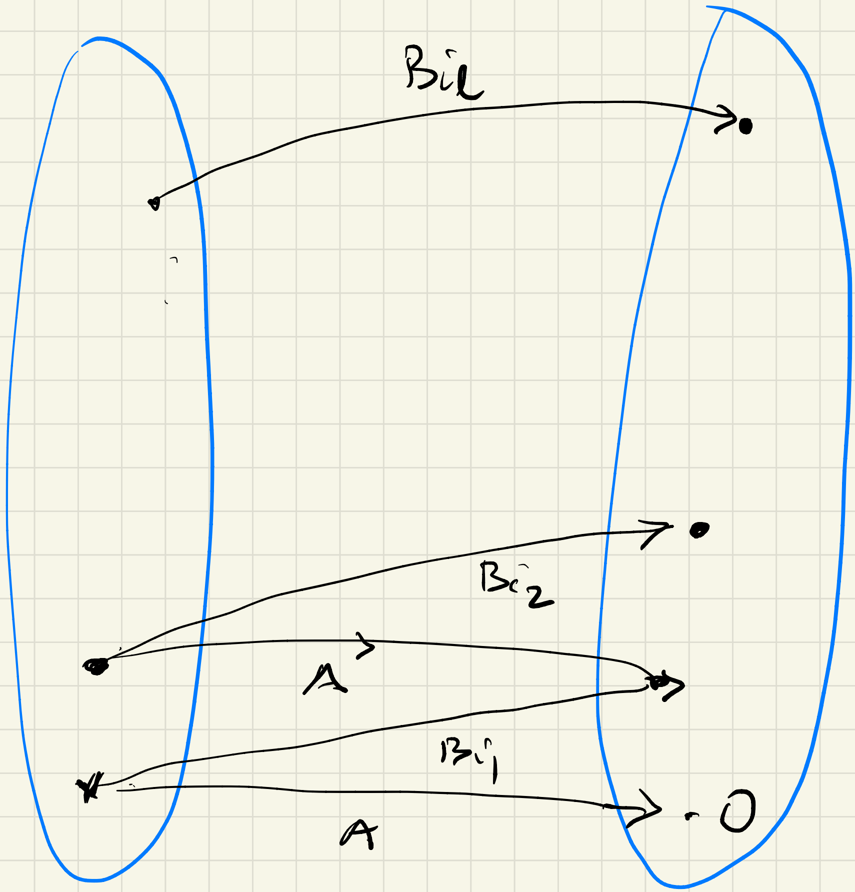
$\therefore$

$A^{-1}(W^*)$ is a $\dim\left(\ker(A)\right)$-singularity witness.

Suggest an augmenting path algorithm:

If $w^* \subseteq \mathrm{Im}(A)$ - done

o.w.:



$$B_{i_\ell} \cdots A^{-1} B_{i_2} A^{-1} B_{i_1} \ker A \notin \mathrm{Im}(A)$$

SIMPLIFY: Assume

$$A = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots 1 & \\ & & & & & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 0 \end{pmatrix}$$

$\therefore \; B_{i_\ell} B_{i_{\ell-1}} \cdots B_{i_1}(v) \notin \mathrm{Im}(A),$

• If $l = 1$,

$$\underbrace{\det\left(xA + B_{i_1}\right)}_{v_1}$$

$$\|$$

$$\det\begin{pmatrix} x & & & & \\ & x & & & \\ & & \ddots & & \\ & & & x & \\ & & & & b \end{pmatrix} = b \cdot x^{n-1} + \cdots$$

Try:

All (n) possible value of $\mathbb{F}$,

Increase

$\uparrow$ the rank;

- If $Bi_1 = Bi_2 \cdots = B$ then

$$B^{\ell} = \begin{pmatrix} * & * \\ * & b \end{pmatrix}$$

- Can choose a Basis

$$\underbrace{Bv,\ B^2v,\ \cdots\cdots}\ \ \underset{\notin \mathrm{Im}(A)}{B^{\ell}v},\ \cdots\cdots\ v$$

$xA+B$ looks like

$$\begin{pmatrix} x & & & * & & & 1 \\ 1 & x & & \vdots & & & 0 \\ & 1 & \ddots & \vdots & & & 0 \\ & & \ddots & * & & & 0 \\ & & & x & * & \vdots & 0 \\ & & & & y & \vdots & 0 \\ & & & & & \ddots & 0 \\ & & & & & * & 0 \\ & & & & & z & 0 \\ & & & 1 & & * & 0 \end{pmatrix}$$

Write $y = x + c$, $z = x + d$, ...

$\det(xA + B) = x^{n-\ell} +$ lower degree terms;

Can you reduce to this case?

$$\underbrace{A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}}$$

$$B = (x_1 B_{i_1} + \cdots + x_\ell B_{i_\ell})^\ell,$$

$$\underbrace{B^\ell(n,n) = b\, x_1 \cdots x_\ell + \text{other summands}}_{\text{degree } \ell}$$

PROBLEM: Interference;

Ex:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\Big\downarrow \quad x^{|y} \text{ by } \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

=

Start with A_1:

$$A_2^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_3^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_2A_3 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$A_3A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$(xA_2+yA_3)^2$ unfortunately $\begin{pmatrix} & & \\ & & \\ & & 0 \end{pmatrix}$

But.....

Skew symm

Work with x, y non commuting;

We may not not be able to ↑ rank in $M(n, \mathbb{F})$ but maybe in an ext;

In this exaple

$$A = \mathbb{R}\langle x, y, z \rangle \big/ (x^2+1, y^2+1, xy-z, yx+z)$$

Then: $$(xA_2 + yA_3)^2 = \begin{pmatrix} z & -1 & \\ 1 & z & \\ & & 2z \end{pmatrix}$$

and: $$A_1 + xA_2 + yA_3 = \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & y \\ -y & -x & 0 \end{pmatrix}$$

HAS FULL RANK;

$$\psi : \mathbb{H} \longrightarrow M_2(\mathbb{C})$$

$$x \longmapsto \begin{pmatrix} i & \\ & -i \end{pmatrix} \quad X$$

$$y \longmapsto \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \quad Y$$

$$z \longmapsto \begin{pmatrix} & i \\ -i & \end{pmatrix} \quad Z$$

Then

$$(A_1 + xA_2 + yA_3) \longmapsto \begin{pmatrix} I & & -X \\ & I & Y \\ -Y & -X & \end{pmatrix}$$

full rank;

Rank has increased not in $\mathcal{B}$ but

$\mathcal{B}^{[2]}$ – second blow up;

- A general phenomenon;

- $B_{i_\ell} \cdots B_{i_1}$ has an entry in (n,n)

- Choose $d > \ell$

$$A' = A \otimes Id$$

$v \in \ker(A)$.

$$\underbrace{B = B_{i_1} \otimes E_{21} + B_{i_2} \otimes E_{32} \otimes \cdots + B_{i_\ell} \otimes E_{\ell+1,\ell}}$$

$\underline{v \otimes u_1} \in \ker(A')$.

- $B^{\ell}(v_1 \otimes u_1)$ has a term

$$B_{i_\ell} \otimes E_{\ell+1,\ell} \circ B_{i_{\ell-1}} \otimes E_{\ell\,\ell-1} \circ \cdots \circ B_{i_1} \otimes E_{21}$$

$$= \underline{B_{i_\ell} \cdots B_{i_1}(v)} \otimes \underline{E_{\ell+1,1}(u_1)}$$

- The second Wong sequence $\langle A', B\rangle$

$w^* \notin \operatorname{Im}(A')$

$\therefore$ $\lambda A' + B$ has rank

$$\geq \underbrace{rd+1} \qquad \underbrace{rd+d} \qquad d(r+1)$$

• Regularity lemma of blow ups:

If $\mathcal{A}$ is a linear subspace of matrices $\operatorname{rk}(\mathcal{A}^{[d]})$ is a multiple of d

Pf: (DM):

$$A = X_0 + t_1 X_1 + \cdots + t_m X_m$$

$$\mathcal{X} = \langle X_1, \ldots X_m \rangle$$

$$\operatorname{rk}(\mathcal{X}^{[d]}) = \operatorname{rk}\left(X_0 \otimes I + X_1 \otimes T_1 + \cdots + X_m \otimes T_m \right)$$

T_i's generic

$$R_d \subseteq \operatorname{Mat}_{d,d}\left(K\left[\{ t^i_{jk} \}\right]\right)$$

Z_d — center of R_d

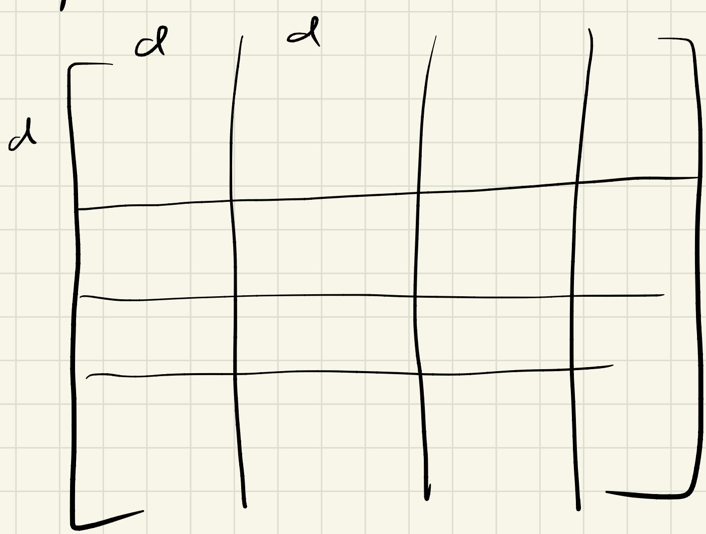
Q_d - field of fractions of Z_d

__AMITSUR:__ $Q_d \otimes_{Z_d} R_d$ is a div alg

Universal div alg of degree d

$X_0 \otimes I + X_1 \otimes T_1 + X_2 \otimes T_2 \cdots + X_m \otimes T_m$

is of form



now do row col operations in $Q_d \otimes R_d$

- Can round up rank from $rd + \Delta$ to $rd + d = (r+1)d$;

Can do this algorithmically in poly time by working in central division algebras;

Thm: Let $\mathcal{B} \leq M(n, \mathbb{F})$; $\mathcal{A} = \mathcal{B}^{[d]}$

Assume $A \in \mathcal{A}$, $rk(A) = rd$;

Let $d' > r$, For large $|\mathbb{F}|$, $\exists$ a det algorithm that returns a $(n-r)d$ shrunk subspace for $\mathcal{A}$ or $A^* \in \mathcal{A} \otimes M(d', F)$ of rank $(r+1)dd'$

DM show null cone is cut of by poly of degree $\leq n^2-n$

<u>Obs</u>: $\mathcal{B}^{[d]}$ contains a nonsing matrix for $d \leq n+1$;

<u>Pf</u>: Assume $d > n+1$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ & & & \\ & & & \\ A_{41} & & & A_{44} \end{bmatrix}$$

Nonsingular.

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ & & & \\ & & & \\ A_{41} & & & A_{44} \end{bmatrix}$$

Nonsingular

rk(blue) ?

$$\begin{bmatrix} & & \\ & & \\ & & A_{44} \end{bmatrix}$$

$$dn - 2n = dn - n + 1 - (n+1)$$

$$> dn - n + 1 - d$$

$$= d(n-1) - (n-1)$$

$$= (d-1)(n-1)$$

Blue $\in \mathcal{B}^{[d-1]}$

$$\therefore \ rk(\text{Blue}) = (d-1)n$$

$\therefore$ Full rank