Basics of algebraic geometry II

\[ \mathbb{C}^{n+1} - \{0\} / \sim =: \mathbb{P}^n \]

Def: \((a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \) if \( \exists \lambda \in \mathbb{C} \setminus \{0\} \) s.t.

\[ b_i = \lambda a_i. \]

Points of \( \mathbb{P}^n \) will be denoted \([a_0 : a_1 : \ldots : a_n]\).  

Eg: \([1 : 2] = [3 : 6] = [1 : 2] \) \( \sim [1 : 2] \subseteq \mathbb{P}^1 \)

\{lines through the origin in \( \mathbb{C}^{n+1} \) \}

We want to talk about zeros of poly in \( \mathbb{C}[x_0, \ldots, x_{n+1}] \)

We first notice that we have to consider only "homogeneous polynomials".

Def: A polynomial \( f \in \mathbb{C}[x_0, \ldots, x_n] \) is homogeneous if every monomial in \( f \) has the same degree.

Eg: \( x_0^2 - x_0 x_1 + x_2 \) homog.
\( x_0^2 - x_0 x_1 + x_2 \) not homog.

\[ f = x_0^2 - x_1 \]
\[ P = [1:2] \]
\[ f(P) = 1 - \frac{1}{2} = 2 - 4 \]
If $f$ is homogeneous of degree $d$, then 
\[ f(a_0, \ldots, a_n) = \lambda^d f(\lambda a_0, \ldots, \lambda a_n) \neq 0 \]
for $\lambda \in \mathbb{C}$, $\lambda \neq 0$.

\[ \therefore \text{whether } f(p) = 0 \text{ or } f(p) \neq 0 \text{ is well-defined.} \]

**Def.** A “projective variety” is a subset of $\mathbb{P}^n$ which is the common set of zeros of a collection of homogeneous polynomials.

Just as in the affine case: we have
\[ \{ \text{radical homog. ideals in } \mathbb{C}[x_0, \ldots, x_n] \} \text{ inclusion reversing bijection } \{ \text{projective algebraic sets in } \mathbb{P}^n \} \]

**Def.** An ideal $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$ is **homog.** if it is generated by homogeneous polynomials.

**Eq.** $(x_1^2 x_2, x_5, x_6 x_1)$ is a homog. ideal.

**Homogeneous Hilbert’s Nullstellensatz:** $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$ homog.
\[ I(\mathbb{Z}(I)) = \sqrt{I} \text{ (radical of } I) \]

**Useful Fact:** Define $U_i : = \{ \mathbb{P}^n | a_i \neq 0 \}$
\[ \text{Zariski open on } \mathbb{P}^n : \text{ closed sets } = \text{ proj alg sets} \]

**Eq.** $U_i = \{ \{ a_0 : a_i \mid a_i \neq 0 \} \subseteq \mathbb{P}^1 \}$
More generally:

More generally: \( V_i \cong C^n \) Later: this is actually an isomorphism of varieties.

Note: \( \mathbb{P}^n = U_0 \cup U_1 \cup \ldots \cup U_n \)

A remark about "varieties":

A 'variety' is either:

- affine variety (i.e., a closed subset in \( \mathbb{A}^n \))
- proj variety (i.e., a closed subset in \( \mathbb{P}^n \))
- quasi-affine variety (i.e., open subset of an affine var)
- quasi-projective variety (i.e., open subset of a projective variety)

\( \mathbb{A}^n - \{0\} \), \( n \geq 2 \) is a variety, but it is not an affine or a projective variety.

\[ a_1 \neq 0 \Rightarrow [a_0 : a_1 : 1] = \left[ \frac{a_0}{a_1} : 1 \right] \text{ in } \mathbb{P}^n \]

\[ V_i = \{ [a_0 : 1] \mid a_0 \in \mathbb{C} \} \]

\[ \text{bij} \quad \longleftrightarrow \quad \mathbb{C} \]
Regular functions:

- $X$ affine variety; $U \subseteq X$ open subset; $p \in U$

A function $f: U \to \mathbb{C}$ is regular at $p$ if

- $\exists g, h \in \mathbb{C}[X_1, \ldots, X_n]$ s.t.
  - $h(p) \neq 0$, and
  - $f = \frac{h}{g}$ on a small neighborhood of $p$ in $U$.

Regular functions = ratios of polynomials.

- We say that $f$ is regular on $U$ if it is regular at all points in $U$.

**Def:** $\mathcal{O}(X) := \{ f: X \to \mathbb{C} \mid f$ regular on $X \}$

**Eq.:**

- $\mathcal{O}(\mathbb{A}^n) = \mathbb{C}[X_1, \ldots, X_n]$

- $X \subseteq \mathbb{A}^n$ affine variety:

$$\mathcal{O}(X) = \frac{k[X_1, \ldots, X_n]}{\mathfrak{I}(X)}$$

There are lots of global regular functions coordinate ring of $X$. 
Def: $X = \mathbb{P}^n$ projective variety; $U \subseteq X$ open, $p \in U$

A function $f: U \rightarrow \mathbb{C}$ is regular at $p$ if

$f$ has a poly of same degree $g, h$ s.t.

$f = \frac{g}{h}$ "locally" at $p$.

Def: $\mathcal{O}(X) := \{ f: X \rightarrow \mathbb{C} \mid f \text{ is reg on } X \}$

Rmk: $\mathcal{O}(\mathbb{P}^n) = \mathbb{C}$; more generally: $\mathcal{O}(X) = \mathbb{C}$ for any projective variety.

Def: $X, Y$ are varieties. A continuous function

$\varphi: X \rightarrow Y$ is a "morphism of varieties" if

for every open set $V \subseteq Y$ and a regular function

$f: V \rightarrow \mathbb{C}$, the function $f \circ \varphi: \varphi^{-1}(V) \rightarrow \mathbb{C}$ is

regular.

Def: $X, Y$ are varieties.

An isomorphism $\psi: X \rightarrow Y$ is a morphism $\psi \circ \psi^{-1} \equiv \text{id}$,

$s.t.$ $\psi \circ \varphi = \text{id} \varphi$,

$\psi \circ \psi = \text{id} \psi$.

Eq: $U_i \xrightarrow{\sim} \mathbb{A}^n$ (with $i$ as above)
Remark: The above is not true for projective varieties.

There is no intrinsic notion of coordinate ring for projective varieties. But we have the following:

Let \( X \subseteq \mathbb{P}^n \) be a projective variety. We have a 

"homogeneous coordinate ring" \( \Gamma(X) := \frac{\mathbb{C}[x_0, \ldots, x_n]}{I(X)} \).

But this depends on the embedding \( X \subseteq \mathbb{P}^n \).

Example: \( \mathbb{P}^1 \hookrightarrow \mathbb{P}^2 \) (2-uple embedding)

\[
\begin{align*}
[a:b] &\mapsto [\sigma^2 : ab : b^2] 
\end{align*}
\]

Then \( \psi_2 \) is an isomorphism \( \mathbb{P}^1 \cong \psi((\mathbb{P}^1) = X \subseteq \mathbb{P}^2 \).

\( \mathbb{P}^1 \rightarrow \mathbb{P}^1 \rightarrow \text{homog coord ring} \quad \mathbb{C}[x_0, x_1] \)

\( \mathbb{P}^1 \rightleftharpoons X \rightarrow \mathbb{P}^2 \rightarrow \text{homog coord ring} \quad \mathbb{C}[x_0, x_1, x_2] \)

\[ \mathbb{C}[x_0, x_1] \nleftarrow \mathbb{C}[x_0, x_1, x_2] \quad \frac{(x_0 x_2 - x_1^2)}{(x_0 x_2 - x_1^2)} \]

Constructible sets, Chevalley’s theorem:

Let \( X \) a variety, a subset \( Y \subseteq X \) is constructible if

it is a finite union of “locally closed” sets.

Def: locally closed = \((\text{open set}) \cap \text{(closed set)}\)

Fact: The set of constructible sets in \(X\) is the smallest
subset of \(P(X)\) containing closed sets which is
closed under finite unions and complements.

Chevalley's thm: \(f: X \to Y\) morphism of varieties

and \(Z \subseteq X\) is constructible

\[ \Rightarrow f(Z) \text{ is constructible.} \]

Eq: Suppose \(G\) acts on \(X\) \((G\ \text{alg gp, } X\ \text{variety})\)

let \(x \in X\). We have \(G \xrightarrow{g} X\)

\[ g \mapsto g \cdot x \]

\[ \text{Im}(g_x) = \text{orbit of } x \text{ is a constructible subset of } X. \]

Fact: For constructible sets, the Euclidean closure is the same as
the Zariski closure.

"Class Variety" \(Y = (y_0, \ldots, y_{l-1})\)

\[ X = (y_0, \ldots, y_k) \]

\[ \overline{X} = (y_0, y_{1,\ldots, k}) \]

\[ V = \text{Sym}^m Y = \text{space of homog forms of deg m in l variables} \]

\[ W = \text{Sym}^n X \]
$G = SL_k(C)$ acts on $V$

$SL_k(C)$ acts on $W$.

We have a map $\phi: W \rightarrow V$

$\phi(w) = y_{0}^{m-n} w$

$W \rightarrow \phi: P(W) \rightarrow P(V)$

$g = \text{det} \ Y \in P V$

$h = \text{perm} \ X \in P W$

$f = \phi(h) \in P V$.

The geometric question:

\[ \Delta_V(f) \neq \Delta_V(g) \]

**Dimension**:

If $X$ is a variety, there is a notion of "dimension" of $X$.

One way: topologically: length of the target chain of irr closed subsets.

Another way: algebraically.

\[ \rightarrow \text{Dim} (A^n) = \text{Dim} (\mathbb{P}^n) = N. \]
→ $U \subseteq X$ open, $X$ irreducible.
\[ \dim U = \dim X. \]

$\Rightarrow \ \ Z(f) \subseteq \mathbb{A}^n \ \ \ \ \ \ \dim Z(f) = n-1.$
$\Rightarrow \ \ Z(f) \subseteq \mathbb{P}^n \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ C[x_1, \ldots, x_n].$

Singularity: $X \subseteq \mathbb{A}^n$ affine variety, $\dim X = r$
$I(X) = (f_1, \ldots, f_t)$
$p \in X.$

"Jacobian matrix": $[\frac{\partial f_i}{\partial x_j}]_{t \times n}$

$X$ is nonsingular at $p$ if $[\frac{\partial f_i}{\partial x_j}(p)]$
has rank $n-r$.

Eg.: $I(X) = (f)$, $\dim X = n-1$.

$X$ is nonsingular at $p$ if
$[\frac{\partial f_i}{\partial x_j}(p) \ldots \frac{\partial f_i}{\partial x_n}(p)]$ has
rank $n-(n-1) = 1$.
\[ \text{Sing}(X) = \{ p \in X \mid X \text{ is singular at } p \} \]

Fact: \( \text{Sing}(X) \) is a proper closed subset of \( X \).