Basics of algebraic geometry II

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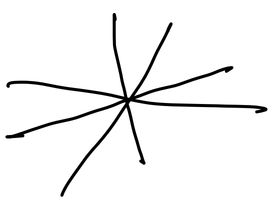
$n \geq 1$ integer; $\mathbb{C}^{n+1} - \{0\} \big/ \sim \; =: \mathbb{P}^n$

<u>Def</u>: $(a_0, .., a_n) \sim (b_0, .., b_n)$ if $\exists\, \lambda \in \mathbb{C} \setminus \{0\}$ s.t
$b_i = \lambda a_i$.

Points of $\mathbb{P}^n$ will be denoted $[a_0 : a_1 : \dots : a_n]$

<u>eg</u>: $[1:2] = [3:6] = [i : 2i] = [\pi : 2\pi] \in \mathbb{P}^1$



$\{$lines through the origin in $\mathbb{C}^{n+1}\}$

$\Updownarrow$

$\mathbb{P}^n$

We want to talk about zeros of poly in $\mathbb{C}[X_0, .., X_{n+1}]$

We first notice that we have to consider <u>only</u> "homogeneous polynomials".

$f = X_0^2 - X_1$

$p = [1:2]$
$= [2:4]$

$f(p) = 1 - 2$
$\neq 2^2 - 4$

<u>Def</u>: A polynomial $f \in \mathbb{C}[X_0, .., X_n]$ is homogeneous if every monomial in f has the same degree.

<u>eg</u>: $X_0^2 - X_0 X_1 + X_2^2$ homog

$X_0^2 - X_0 X_1 + X_2$ not homog

If f is homogeneous of degree d, then

$$f(a_0, .., a_n) = \lambda^d f(\lambda a_0, \ldots, \lambda a_n) \quad \forall \lambda \in \mathbb{C}, \lambda \neq 0.$$

$\therefore$ whether $f(p) = 0$ or $f(p) \neq 0$ is well-defined.

<u>Def</u>: A "projective variety" is a subset of $\mathbb{P}^n$ which is the common set of zeros of a collection of homogeneous poly.

Just as in the affine case: we have

$$\left\{ \begin{array}{c} \text{radical homog} \\ \text{ideals in } \mathbb{C}[x_0, .., x_n] \end{array} \right\} \xleftrightarrow[\text{bijection}]{\text{inclusion reversing}} \left\{ \begin{array}{c} \text{projective algebraic} \\ \text{sets in } \mathbb{P}^n \end{array} \right\}$$

<u>Def</u>: An ideal $I \subseteq \mathbb{C}[X_0, .., X_n]$ is <u>homog</u> if it is generated by homogeneous poly.

<u>eg</u>: $(X_1^2 X_2, X_0^5, X_6 X_1)$ is a homog ideal

<u>Homogeneous Hilbert's Nullstellensatz</u>: $I \subseteq \mathbb{C}[X_0, .., X_n]$ homog

$$I(Z(I)) = \sqrt{I} \quad (\text{radical of } I)$$

<u>Useful fact</u>: Define $U_i := \{ [a_0 : \cdots : a_n] \in \mathbb{P}^n \mid a_i \neq 0 \}$

$U_i \subseteq \mathbb{P}^n$

Zariski top on $\mathbb{P}^n$: closed sets = proj alg sets

<u>eg</u>: $U_1 = \{ [a_0 : a_1] \mid a_1 \neq 0 \} \subseteq \mathbb{P}^1$

$a_1 \neq 0 \Rightarrow [a_0 : a_1] = [\frac{a_0}{a_1} : 1]$ in $\mathbb{P}^n$

$$U_1 = \{ [a_0 : 1] \mid a_0 \in \mathbb{C} \} \xleftrightarrow{\text{bijection}} \mathbb{C}$$

More generally: $U_i \xleftrightarrow{\text{bij}} \mathbb{C}^n$, $U_i \subseteq \mathbb{P}^n$

Later: this is actually an isomorphism of varieties.

Note: $\mathbb{P}^n = U_0 \cup U_1 \cup \dots \cup U_n$, where $U_0 \text{ is } \mathbb{A}^n$, $U_1 \text{ is } \mathbb{A}^n$, $U_n = \mathbb{A}^n$

A remark about "varieties".

A 'variety' is either:

- affine variety (i.e., a closed subset in $\mathbb{A}^n$)
- proj variety (i.e., a closed subset in $\mathbb{P}^n$)
- quasi-affine variety (i.e., open subset of an affine var)
- quasi-projective variety (i.e., open subset of a projective variety)

eg: $\mathbb{A}^n - \{0\}$, $n \geq 2$ is a variety; but it is not an affine or a projective variety.

Regular functions:

$X \subseteq \mathbb{A}^n$ closed

- X affine variety ; $U \subseteq X$ open subset ; $p \in U$

A function $f: U \to \mathbb{C}$ is regular at p if $\exists\, g, h \in \mathbb{C}[X_1, .., X_n]$ s.t.

- $h(p) \neq 0$, and
- $f = \frac{h}{g}$ on a small neighborhood of p in U.

Regular functions = ratios of polynomials.

- We say that f is regular on U if it is regular at all points in U

Def: $\mathcal{O}(X) := \{ f: X \to \mathbb{C} \mid f \text{ regular on } X \}$

eg:
- $\mathcal{O}(\mathbb{A}^n) = \mathbb{C}[X_1, .., X_n]$
- $X \subseteq \mathbb{A}^n$ affine variety:

$$\mathcal{O}(X) = \frac{k[X_1, .., X_n]}{I(X)} = \Gamma(X)$$

$\Gamma(X)$: coordinate ring of X.

There are lots of global regular functions

Def: $X \subseteq \mathbb{P}^n$ projective variety; $U \subseteq X$ open, $p \in U$

A function $f : U \to \mathbb{C}$ is regular at p if $\exists$ homog poly of same degree g, h s.t

$$f = \frac{g}{h} \text{ "locally" at } p.$$

Def: $\mathcal{O}(X) := \{ f : X \to \mathbb{C} \mid f \text{ is reg on } X \}$

Rmk: $\mathcal{O}(\mathbb{P}^n) = \mathbb{C}$; more generally: $\mathcal{O}(X) = \mathbb{C}$ for any projective variety.

Def: X, Y are varieties. A continuous function $\varphi : X \to Y$ is a "<u>morphism of varieties</u>" if for every open set $V \subseteq Y$ and a regular function $f : V \to \mathbb{C}$, the function $f \circ \varphi : \varphi^{-1}(V) \to \mathbb{C}$ is regular.

$$\begin{tikzcd} \varphi^{-1}(V) \arrow[r, "\varphi"] \arrow[rr, bend right, "f \circ \varphi"'] & V \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

Def: X, Y are varieties.

An isomorphism $\varphi : X \to Y$ is a morphism s.t $\exists$ a morphism $\psi : Y \to X$ s.t $\varphi \circ \psi = id_Y$

$\psi \circ \varphi = id_X$.

Eg: $U_i \xrightarrow[\text{iso of varieties}]{\sim} \mathbb{A}^n$ (U_i as above)

Rmk: The above is not true for projective varieties.

There is no intrinsic notion of coordinate ring for projective varieties. But we have the following:

Let $X \subseteq \mathbb{P}^n$ be a projective variety. We have a "homogeneous coordinate ring" $\Gamma(X) := \dfrac{\mathbb{C}[x_0, .., x_n]}{I(X)}$.

But this depends on the embedding $\boxed{X \subseteq \mathbb{P}^n}$

eg: $\mathbb{P}^1 \xrightarrow{\varphi_2} \mathbb{P}^2$ (2-uple embedding)

$[a:b] \mapsto [a^2 : ab : b^2]$

Then φ_2 is an isomorphism $\mathbb{P}^1 \xrightarrow{\sim} \varphi(\mathbb{P}^1) = X \subseteq \mathbb{P}^2$

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \rightsquigarrow$ homog coord ring $\mathbb{C}[x_0, x_1]$

$\mathbb{P}^1 \xrightarrow{\sim} X \hookrightarrow \mathbb{P}^2 \rightsquigarrow$ homog coord ring $\dfrac{\mathbb{C}[x_0, x_1, x_2]}{I(X)}$

$$\mathbb{C}[x_0, x_1] \not\cong \frac{\mathbb{C}[x_0, x_1, x_2]}{(x_0x_2 - x_1^2)} \qquad \frac{\mathbb{C}[x_0, x_1, x_2]}{I(X)} = \frac{\mathbb{C}[x_0, x_1, x_2]}{(x_0x_2 - x_1^2)}$$

Constructible sets, Chevalley's thm:

X a variety; a subset $Y \subseteq X$ is constructible if it is a finite union of "locally closed" sets.

Def: locally closed = (open set) $\cap$ (closed set)

Fact: The set of constructible sets in X is the smallest subset of $P(X)$ containing closed sets which is closed under finite unions and complements.

Chevalley's thm: $f: X \to Y$ morphism of varieties and $Z \subseteq X$ is constructible
$\Rightarrow f(Z)$ is constructible.

Eg: Suppose G acts on X (G alg gp, X variety)
Let $x \in X$. We have
$$G \xrightarrow{\varphi_x} X$$
$$g \mapsto g.x$$
$\mathrm{Im}(\varphi_x) =$ orbit of x is a constructible subset of X.

Fact: For constructible sets, the Euclidean closure is the same as the zariski closure.

"Class variety"

$Y = (y_0, \ldots, y_{\ell-1})$

$X = (y_1, \ldots, y_k)$

$\overline{X} = (y_0, y_1, \ldots, y_k)$

Let $m > n$;
$\ell = m^2$
$k = n^2$

$V = \mathrm{Sym}^m Y$ = Space of homog forms of deg m in ℓ variables

$W = \mathrm{Sym}^n X$

$G = SL_l(\mathbb{C})$ acts on V — deg n forms

$SL_k(\mathbb{C})$ acts on W. — deg m forms

$\underline{\underline{m > n}}$

We have a map $\phi : W \hookrightarrow V$

$w \longmapsto y_0^{m-n} w$

$\rightsquigarrow \phi : \mathbb{P}(W) \hookrightarrow \mathbb{P}(V)$

$g = \det(Y) \in \mathbb{P}V$

$h = \mathrm{perm}(X) \in \mathbb{P}W$

$f = \phi(h) \in \mathbb{P}V$.

"class variety" of g = closure of the orbit of g under SL_l action. $\Delta_V(g)$.

The geometric question : $\boxed{\Delta_V(f) \not\subseteq \Delta_V(g)}$

Dimension :

If X is a variety, there is a notion of "dimension" of X.

one way : topologically : length of the largest chain irr closed subsets.

another way : algebraically.

$\rightarrow \mathrm{Dim}(\mathbb{A}^n) = \mathrm{Dim}(\mathbb{P}^n) = n$.

$\rightarrow$ $U \subseteq X$ open, X irr

$\dim U = \dim X$.

$\rightarrow$ $\left.\begin{array}{l} Z(f) \subseteq \mathbb{A}^n \\ Z(f) \subseteq \mathbb{P}^n \end{array}\right\}$ $\dim Z(f) = n-1$.

$\mathbb{C}[X_1, \dots, X_n]$

<u>Singularities</u>: $X \subseteq \mathbb{A}^n$ affine variety; $\dim X = r$

$I(X) = (f_1, \dots, f_t)$ $\qquad p \in X$.

"Jacobian matrix": $\left[\dfrac{\partial f_i}{\partial X_j}\right]_{t \times n}$

X is <u>nonsingular at p</u> if $\left[\dfrac{\partial f_i}{\partial X_j}(p)\right]$

has rank $n-r$.

<u>Eg</u>: $I(X) = (f)$, $\dim X = n-1$.

X is <u>nonsingular at p</u> if

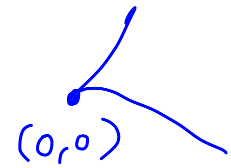
$\left[\dfrac{\partial f}{\partial X_1}(p) \quad \cdots \quad \dfrac{\partial f}{\partial X_n}(p)\right]$ has

rank $n-(n-1) = 1$

$\therefore$ X is nonsing at $P \iff \dfrac{\partial f}{\partial x_i}(P) \neq 0$ for some $1 \leq i \leq n$.

<u>Eg</u>: $(0,0)$ is a singular point of $X = Z(x_1^2 - x_2^3)$

only singularity is $(0,0)$.



<u>Eg</u>: $Z(x_1^2 - x_2)$ is nonsingular.

"Singular locus of X"

$$\mathrm{Sing}(X) = \{ P \in X \mid X \text{ is singular at } P \}$$

<u>Fact</u>: $\mathrm{Sing}(X)$ is a proper closed subset of X.