# Exp. Deg. Lower bounds on invariant rings.

(Adv. Math, 2020 j/w Harm Derksen)

## Motivation:

$G \curvearrowright V$ [gp reductive, V rational]

$G = SL_n \times SL_n \times SL_n \curvearrowright V = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$

$\mathbb{C}[V]$ = polynomials in $V$

[$V = \mathbb{C}^n$, $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]$]

$$\mathbb{C}[V]^G = \{ f \in \mathbb{C}[V] \mid f(gv) = f(v) \ \forall g \in G, v \in V \}$$

## Landmark Results:

① Invariant rings ($\mathbb{C}[V]^G$) are finitely generated.

Hilbert

Questions:

① What are the generators?

①' Minimal set of generators

② Relations among generators.

→ $19^{th}$ century

Realize over several decades →

- Can do this for some small cases
  moderate cases

- It's pretty difficult in general.

→ ⟹ Complexity theory & Computers

Natural: How hard to compute?

[GCT V: Mulmuley.]

A naive measure of computational hardness is degree bounds.

Problem [Deg. bds]

Give strong bounds for

$$\beta(G, V) = \min \{ d \mid \mathbb{C}[V]^G_{\leq d} \text{ generate } \mathbb{C}[V]^G \}$$

Complexity aside $\to$ deg $d$ invariants are computable.

but without a degree bound, cannot get an algorithm naively to compute generators.

Hilbert: $\beta(G,V) < \infty$

Popov 80's: $\beta(G,V) \sim 2^{2^n}$

Derksen 2000: $\beta(G,V) \sim 2^{\mathrm{poly}(n)}$

[Derksen, M. 2015]: L-R action

$$\underbrace{SL_n \times SL_n}_{=G} \curvearrowright \underbrace{\mathrm{Mat}_{n,n}^m}_{=V} \quad \left[ (\mathbb{C}^n \otimes \mathbb{C}^n)^{\oplus m} \right]$$

$\beta(G,V) \le n^6$

$\Rightarrow$ RIT $\in$ P

IQS $\Rightarrow$

$$\det \begin{pmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{pmatrix}$$

$\to$ each block is a lin. comb of $X_1 \to X_m$

L-R action $\longrightarrow$ RIT

$G \curvearrowright V \rightsquigarrow$ Null Cone Membership (NCM)

Naive Hope: Better deg. bds $\Rightarrow$ faster algos for NCM

Next generalization of the L-R action is

$G = SL_n \times SL_n \times SL_n \curvearrowright V = (\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n)^{\oplus m}$

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Thm [Derksen, M.]

$$\beta(SL_{3n}^{\times 3}, (\mathbb{C}^{3n} \otimes \mathbb{C}^{3n} \otimes \mathbb{C}^{3n})^{\oplus 5}) \geq 4^n - 1$$

Thm [Derksen, M.]

$SL_{3n} \curvearrowright (\underbrace{Sym^3(\mathbb{C}^{3n})}_{\text{deg 3 polys in } 3n \text{ variables.}})^{\oplus 4}$

$\boxed{\beta \geq \frac{2}{3}(4^n - 1).}$ $\longrightarrow$ Indicate a pf for.

$\mathbb{C}^{3n} \longrightarrow$ basis $\left\{ \begin{matrix} x_1 - - x_n \\ y_1 - y_n \\ z_1 - - z_n \end{matrix} \right\}$

$\mathrm{Sym}^3 \mathbb{C}^{3n} =$ deg 3 polys in $\{x_i, y_i, z_i\}_{1 \le i \le n}$

$\circlearrowright$

$SL_{3n}$

<u>Idea</u>: Lifting l.bds from torus actions.

# Grosshans Principle

$G \curvearrowright V, W$ $\quad v \in V$ closed orbit ($G.v$ is a closed subset).

$$G_v = \{g \in G \mid gv = v\} = H.$$

$$\boxed{\mathbb{C}[V \oplus W]^G \twoheadrightarrow \mathbb{C}[W]^H}$$

Cor: $\beta(G, V \oplus W) \geq \beta(H, W)$.

$$V \times W /\!/ G \hookleftarrow G.v \times W /\!/ G$$

$$\left[ G.v \simeq G/H \right]$$

$$G.v \times W /\!/ G = W /\!/ H$$

$w_1, w_2$ in the same $H$-orbit

$\Updownarrow$

$(v, w_1)$ & $(v, w_2)$ in the same $G$-orbit.

$G \curvearrowright V, W$

$v \in V$ closed orbit

$G_v = \{g \in G \mid gv = v\}$

$$\beta(G, V \oplus W) \geq \beta(H, W).$$

Steps:

1. Identify a $v \in V$ with closed orbit
2. Try and ensure $H$ is a torus
3. Ability to prove l.bd for $H$-action.

Challenge: How to prove that $v \in V$ has a closed orbit.

4. Getting all the steps to work at once.

Torus actions:

$T = (\mathbb{C}^*)^k \curvearrowright V = \mathbb{C}^m$ → coordinates are $x_1, \ldots, x_m$

weight matrix

$$W = \begin{bmatrix} | & & | \\ w_1 & \cdots & w_m \\ | & & | \end{bmatrix} \qquad w_i \in \mathbb{Z}^k.$$

Invariant polynomials

$\mathrm{Ker}(W) \cap \mathbb{Z}^m_{\geq 0}$ ⟶ Inv. monomial

$\Psi$

$(a_1, \ldots, a_m)$ ⟶ $x_1^{a_1} \cdots x_m^{a_m}$

$\sum a_i w_i = 0$

→ Semi-group.

Generators ⟷ generators.

Eg: Rig things so that

$$W = \begin{bmatrix} 1 & -2 & & \\ & 1 & -2 & \\ & & 1 & -2 \end{bmatrix}$$

$$\mathrm{Ker}(W) = \langle (8, 4, 2, 1) \rangle$$

Extending pattern

$(2^n, 2^{n-1}, 2^{n-2}, \ldots, 1)$

$\downarrow$ corr. monomial

$x_1^{2^n} x_2^{2^{n-1}} \cdots x_n$

# Hard part: Closed Orbit.

$\beta(SL_{3n}, \mathrm{Sym}^3(\mathbb{C}^{3n})^{\oplus 4}) \geq \frac{2}{3}(4^n - 1)$

$G = SL_{3n}$. $\quad V = \mathrm{Sym}^3(\mathbb{C}^{3n})^{\oplus 3}$

$W = \mathrm{Sym}^3(\mathbb{C}^{3n})$

$$v = \Big(\sum_i x_i^2 z_i,\ \sum_i y_i^2 z_i,\ \sum 2^i x_i y_i z_i\Big)$$

$\downarrow$

$\mathrm{Sym}^3 \mathbb{C}^{3n}$

① Orbit of $v$ is closed.

② $T = (\mathbb{C}^*)^n \curvearrowright \mathbb{C}^{3n}$

$(t_1, \dots, t_n) \cdot x_i = t_i x_i$

$y_i = t_i y_i$

$z_i = t_i^{-2} z_i$

Stabilizer of $v$ i.e $G_v = H$

$\rho : (\mathbb{C}^*)^n \longrightarrow SL_{3n}$

$\mathrm{Im}(\rho) = G_v$

$\boxed{③\ \beta(T = H, \mathrm{Sym}^3(\mathbb{C}^{3n})) \geq \frac{2}{3}(4^n - 1)}$

Why is the $SL_{3n}$-orbit of

$$v = \left(\sum x_i^2 z_i,\ \sum y_i^2 z_i,\ \sum 2\, x_i y_i z_i\right)$$ closed?

replace with numbers that have abs. value 1.

<u>Idea</u>: Moment map $\mu$

<u>Kempf-Ness</u>:

$\mu(v) = 0 \implies$ Orbit of $v$ is closed.

$\Downarrow$

v is a point of min'l norm in the orbit $\Big\} \to$ Such a point of min'l norm exists if and only if the orbit is closed

Check that $\mu(v) = 0$.

How exactly to check $\mu(v)=0$.

$$\mu : V \longrightarrow \mathrm{Lie}(G)^*$$
$$v \longmapsto (X \longrightarrow \langle Xv, v\rangle)$$

Dadok-Kac Criterion:

$V = \bigoplus_{\lambda \in I} V_\lambda$ wt. space decomposition

$v \in V$ $\quad$ $\mathrm{supp}(v) = \{\lambda \mid v_\lambda \neq 0\}$

$= (v_\lambda)_{\lambda \in I}$

1. $\mathrm{conv}(\mathrm{supp}(v)) \ni 0$ in the interior
2. $\alpha, \beta \in \mathrm{supp}(v) \Rightarrow \alpha - \beta$ is not a root.

$1 + 2 \Rightarrow \mu(v) = 0$.

Baby version $SL_3 \curvearrowright \mathrm{Sym}^3 \mathbb{C}^3$

$v = x^3 + y^3 + z^3$

$\mathrm{Supp}(v) = \{(3,0,0), (0,3,0), (0,0,3)\}$ $\begin{Bmatrix} \text{modulo} \\ \langle(1,1,1)\rangle \end{Bmatrix}$

① $\mathrm{conv}(\mathrm{supp}(v)) \ni 0$ in the interior is okay

② Roots are $\varepsilon_i - \varepsilon_j$

$\{\pm(1,-1,0), \pm(1,0,-1), \pm(0,1,-1)\}$

So ② is fine.

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<u>Qn:</u> $h_3(x,y,z) = x^3+y^3+z^3+x^2y+xy^2$
$+xz^2+x^2z+y^2z+yz^2$
$+xyz$

Is the $SL_3$-orbit closed?

$SL_n$ actions (polynomial)

$\to$ deg $d$ poly actions

$d \geq 3 \to$ exp. lbds

$d = 2 \to$ ?

$d = 1 \to$ trivial

$\to SL_n \curvearrowright Mat_{n,n}^{n^2}$

$g \cdot (A_1, \ldots, A_{n^2}) = (gA_1g^T, \ldots, gA_{n^2}g^T)$

<u>Qn</u>: Is the deg. bd for this action polynomial or exponential?