Affine vs Projective Orbit Closures, Lie Algebra Actions and Constructions [1]

GCT2022 WORKSHOP

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[1] Background for the joint work with Milind Sohoni and K V Subrahmanyam GCT - Lie Algebraic Methods for Projective Limits of Stable Points.

Outline

- Group actions
  - Affine vs projective orbits
  - An example
- Induced Lie algebra actions
  - A geometric view
  - Examples
- The key questions
- Luna slice theorem for a stable orbit nbd
- Conjugation action
  - Affine orbit closures
  - Projective orbit closures
  - The main theorem
- The local model
  - The associated bundle
  - The quotient construction
  - The key choices
- Preview of the next talk

Group actions: affine vs projective orbits

Group actions

G - a linear *algebraic* group over $k = \mathbb{C}$ such as GL_n, SL_n
G acts linearly on a vector space V via $\cdot : G \times V \to V$

$$(g, v) \mapsto g \cdot v$$

The associated representation: $\rho : G \to GL(V)$

Example: $GL_1 = k^*$ acts on $V = k^2$ via representation

$$\rho(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$$

- Induced action on the ring $k[V] = k[x, y]$ of polynomials :

$$t.x = tx \quad t.y = t^{-1}y$$

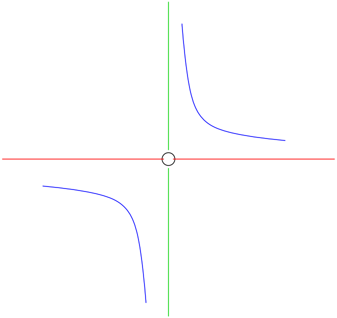
- The ring $k[V]^G$ of invariants is generated by xy.

$$k[V]^G = k[xy] \subset k[x, y]$$

Affine orbits

General set-up: $\rho : G \to GL(V)$

Example: $\rho(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$



Affine orbits:

- Orbit of p: $O(p) = \{g \cdot p \mid g \in G\}$
- Stabilizer of p: $G_p = \{g \in G \mid g \cdot p = p\}$
- p is stable if $O(p)$ is closed in V (Zariski topology)

Projective orbits

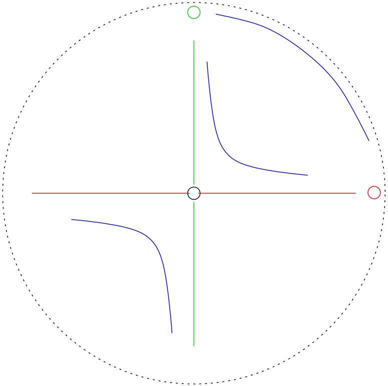
- General set-up: G acts linearly on V
- Induced action of G on $P(V)$ - the set of lines of V through origin. For an element $p \neq 0 \in V$, $[p] \in P(V)$
- Projective orbits:
  - Projective orbit $O([p]) = \{g \cdot [p] | g \in G\} \subseteq P(V)$
    This can be seen as a cone over the affine orbit $O(p)$
  - Projective stabilizer of $[p]$: $G_{[p]} = \{g \in G \mid g \cdot [p] = [p]\}$

$$G_p \subseteq G_{[p]}$$

    The new group elements simply translate p on the line [p]
  - $O([p])$ is almost always non-closed! (even when p is stable)

Revisit:

$$\rho(t) = \left[\begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right]$$



Lie algebra actions/representations

Induced Lie Algebra Actions

The group action induces a Lie algebra action – its infinitesimal version.

- $\mathcal{G}$ = the tangent space to G at identity e.
  Lie bracket $[,] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ satisfying Jacobi identity ....
- Lie algebra of the group $GL(V)$
  $gl(V) = End(V) =$ all linear transformations on V.
  The Lie bracket is $[X, Y] = XY - YX$.
- $\rho : G \to GL(V)$ induces a Lie algebra representation

  $$d\rho_e : T_e G = \mathcal{G} \to T_e GL(V) = gl(V)$$

  This is simply the derivative of ρ at the identity.

Induced Lie Algebra Actions: A geometric view

- The Lie algebra representation $\rho' : \mathcal{G} \to End(V)$
- The associated Lie algebra action $@ : \mathcal{G} \times V \to V$
- For a point p and an infinitesimal group element $\mathfrak{g}$,

$$\mathfrak{g}@p = \rho'(\mathfrak{g})p = \text{direction along which } \mathfrak{g} \text{ moves point } p$$

- $\mathfrak{g}@p$ is bi-linear; linear both in $\mathfrak{g}$ and p.
- $\mathfrak{g}$ produces a smooth vector field on V
- Non-linear action: Let G act on M - a manifold/variety
  - Group action $\cdot : G \times M \to M$. Each $\mathfrak{g} \in \mathcal{G}$ produces a smooth vector field ($\mathfrak{g}@m \in T_m M$) on M – in notation: $\mathfrak{g} \rightsquigarrow X_{\mathfrak{g}}$.
  - Let $Vec(M)$ be the set of smooth vector fields on M. This is a Lie algebra with the natural Lie bracket of vector fields on M
  - The Lie algebra action $\rightsquigarrow: \mathcal{G} \to Vec(M)$ is a Lie algebra homomorphism

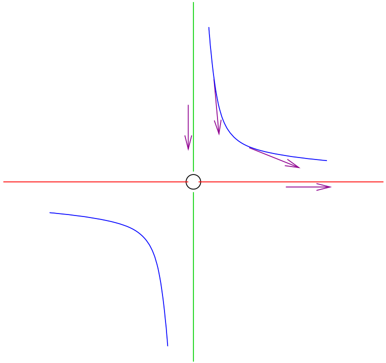
$$[\mathfrak{g}_1, \mathfrak{g}_2] \rightsquigarrow [X_{\mathfrak{g}_1}, X_{\mathfrak{g}_2}]$$

Example ..

Our linear action $\cdot : k^* \times k^2 \to k^2$.

$$\rho(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \quad \rho'(1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that ρ' is a linear map $k \to End(k^2)$



The vector field at $p = (\alpha, \beta)$ points in the direction $(\alpha, -\beta)$.
This is the tangent to the orbit at p.

Action on forms

GL_n acts on $Sym^d(X)$ - degree d forms in n variables X
For $g = (a_{ij}) \in GL_n, \mathfrak{g} = (b_{ij}) \in gl_n$ and $f \in Sym^d(X)$

$$g.f(x_1,\ldots,x_n) = f(a.\mathbf{x}) = f(\ldots,\sum_j a_{ij}x_j,\ldots)$$

$$\mathfrak{g}@f = \sum_i \sum_j b_{ij} x_i \frac{\partial f}{\partial x_j}$$

Example: let $\mathfrak{g} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $f = x^2 + xy$:

$$\begin{aligned} \mathfrak{g}@f &= x\tfrac{\partial f}{\partial x} + x\tfrac{\partial f}{\partial y} + y\tfrac{\partial f}{\partial x} \\ &= x(2x+y) + x(x) + y(2x+y) \\ &= 3x^2 + 3xy + y^2 \end{aligned}$$

Conjugation action: adjoint representation

GL_m acts on M_m - $m \times m$ matrices

$$\begin{array}{rl} GL_m \times V \to V & (g, x) \mapsto gxg^{-1} \\ gl_m \times V \to V & (\mathfrak{g}, x) \mapsto [g, x] = gx - xg \end{array}$$

Example: let $\mathfrak{g} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\begin{aligned} \mathfrak{g}@x &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

In general, the group G action induces the Lie algebra $\mathcal{G}$ action

- For $p \in V$, the subspace $W = \{\mathfrak{g}@p \mid \mathfrak{g} \in \mathcal{G}\}$ is the tangent space to the orbit $O(p)$ at p.
- Stabilizer Lie subalgebra $\mathcal{G}_p = \{\mathfrak{g} \mid \mathfrak{g}@p = 0\}$ is the Lie algebra associated to the stabilizer subgroup G_p
- The action at p induces the linear map $\mathcal{G} \to V$ $(\mathfrak{g} \mapsto \mathfrak{g}@p)$ with kernel $\mathcal{G}_p$ and image - the tangent space to the orbit. $@p : \mathcal{G} \to V$ results in the identification $\mathcal{G}/\mathcal{G}_p \cong T_p O(p)$

Group actions	Lie algebra actions
Global	Local
Orbit	Tangent-space to the orbit
Stabilizer subgroup	Stabilizer Lie subalgebra [2]

[2] determines the *connected* component of the stabilizer subgroup

The Key Questions - Orbit Closures

Given p and q in V,

- When is $p \in \overline{O(q)}$? (affine orbit closure membership)
- When is $[p] \in \overline{O([q])}$? (projective orbit closure membership)

Algebraic complexity theory, GCT settings

- Determinant vs Permanent question: projective orbit closure question (p = padded-permanent and q=determinant)
- The p and q of interest in GCT are completely determined by their stabilizers
- The point q=determinant is, in fact, stable for SL-action and p=permanent is partially-stable...

Action in neighborhood

Neighorhood questions

- Given p, which orbits come arbitrary close to p?
- How are stabiliziers of points in the neighborhood of p related to G_p - the stabilizer of p?

These questions are answered when p is stable
In this case, the stabilizer G_p is reductive

Kempf ..

If $p \in \overline{O(q)}$, there is an an optimal 1-PS $\lambda : k^* \rightarrow G$ which drives q to $O(p)$, that is, $\lim_{t\rightarrow 0} \lambda(t).q \in O(p)$. It also aligns G_q inside G_p.

Luna

Luna's slice theorem describes a tubular neighborhood of the entire orbit $O(p)$. If q is in this neighborhood, a conjugate of G_q is a subgroup of G_p.

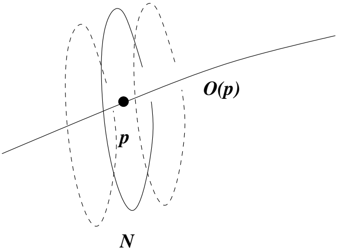
Luna model - the underlying space - associated bundle

$O(p)$ - the closed orbit and $H = G_p$ the reductive stabilizer

- Write $V = T_pV = T_pO(p) \oplus N$ where N is a normal *slice* – H-invariant complement to the tangent space to the orbit.
- H acts on $G \times N$ as

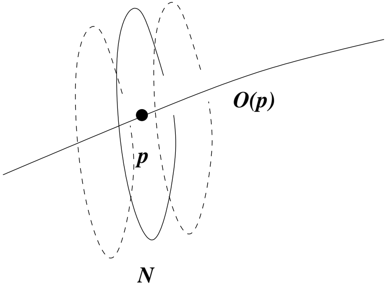
$$h \cdot (g, n) = (gh^{-1}, h.n)$$

The H-quotient is $G \times_H N$; $[(g, n)]_H$ is the H-orbit of (g, n).



- $G \times_H N$ is the normal bundle of the orbit $O(p) = G/H$ in V; It is a twisting of N along G/H and will serve as the model

Luna model - the G-action on the model $G \times_H N$



- G also acts on $G \times N$

$$g' \cdot (g, n) = (g'g, n)$$

As this action commutes with the H-action on $G \times N$, we have G-action on $G \times_H N$:

$$g'.[(g, n)]_H = [(g'.g, n)]_H$$

So, an explicit description of G-action on $G \times_H N$.
This is purely synthetic and we only need H-action on N.

Relating the model to the ambient space V

$O(p)$ - closed, $H = G_p$ reductive and N - a normal H-complement

- Commuting actions of H and G on $G \times N$

$$h \cdot (g, n) = (gh^{-1}, h.n) \qquad g' \cdot (g, n) = (g'g, n)$$

  The H-quotient is $G \times_H N$; G acts on $G \times_H N$

- The map $\mu : G \times N \to V$ defined as $\mu(g, n) = g \cdot (p + n)$ is G-equivariant and constant on H-orbits

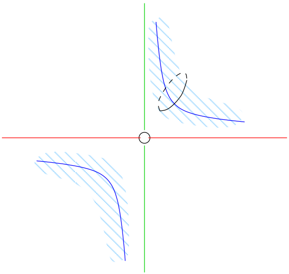
$$\mu(gh^{-1}, h.n) = gh^{-1}(p + h.n) = g.(p + n)$$

  So, μ descends to a G-equivariant map $\phi : G \times_H N \to V$.

$$\phi([(g, n)]_H) = g.(p + n)$$

Luna Slice Theorem

The map ϕ sets up an G-isomorphism between an invariant nbd of G/H in $G \times_H N$ and an invariant nbd of $O(p)$.



Luna Slice Theorem:

- Applicable to only stable points.
- Provides a tubular-neighborhood of the entire orbit
- A description of the G-action in this nbd can be factored via an explicit G-action on $G \times_H N$

Objective

To develop a Lune-type factorizable model of a nbd of any p.

Central difficulty: $H = G_p$ may not be reductive and hence a H-invariant normal slice may not exist.

Lie algebraic local model

Our result

- We develop a Luna-type model which applies to all points.
- It is local - we restrict to a nbd of the point and not the entire orbit, and work with Lie algebra action in this nbd.
- We provide an explicit description of the Lie algebra action on the model.

Application of local model

We apply the local model to the conjugation action to understand orbits in the vicinity of nilpotent orbits.

One Matrix under Conjugation

Conjugation action

$G = GL_m$ acts on V - the space of $m \times m$ matrices $g.A = gAg^{-1}$

- Orbits are parametrized by Jordan canonical forms
- Invariant ring generated by coefficients of the characteristic polynomial
$$det(A - \lambda I)$$
these coefs are symmetric functions of eigenvalues.
- Stable points = affine closed orbits = diagonalizable matrices

Affine Closed Orbits

Parametrized by the numerical data: eigenvalues along with their multiplicites.

Orbits in the null-cone and their closures

- Null-cone = the nilpotent matrices
- The nilpotent matrix J_a is a single Jordan $a \times a$ block

$$J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Any nilpotent matrix has a Jordan canonical form $J_{\bar{a}} = J_{a_1} \oplus J_{a_2} \oplus \ldots \oplus J_{a_s}$ where $\bar{a} = (a_1 \geq a_2 \geq \ldots \geq a_s)$ is a partition of m

Orbits in the null-cone

Parametrized by the combinatorial data: partitions of m which record the sizes of the nilpotent blocks

Affine orbit closures

Gerstenhaber-Hesselink Theorem for orbit-closures in the nullcone

$J_{\bar{a}}$ is in the affine orbit closure of $J_{\bar{b}}$ iff $\bar{a} \trianglelefteq \bar{b}$, that is,

$$a_1 \leq b_1, a_1 + a_2 \leq b_1 + b_2, \ldots.$$

Nilpotent-orbit-closures are given by dominance order on partitions.

In particular, Null-cone = orbit closure of J_m.
These orbit closures are also projective orbit closures.

A similar result for affine orbit closures of arbitrary matrices;
However the numerical data (that is, eigenvalues with multiplicities) are preserved in the affine orbit closure.
The orbit of a Jordan canonical form contains in its affine closure the *closed* orbit of the diagonal part.

Projective orbit closures

Consider $y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the 1-PS $\lambda(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$

$$\begin{aligned}
\lambda(t) y \lambda(t^{-1}) &= \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix} \\
&= \begin{bmatrix} 0 & t^2 \\ t^{-2} & 0 \end{bmatrix} = t^{-2} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + t^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= t^{-2} \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + t^4 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \\
&= t^{-2}(n + t^4 n') \sim n + t^4 n'
\end{aligned}$$

The projective orbit closure of the stable point y contains the nilpotent matrix n.

The projective orbit closure question

- x - a nilpotent matrix given by its nilpotent-block partition.
- Let y be any matrix described by its Jordan canonical form.
- When does $[x] \in \overline{O([y])}$?

Local model at J_m

- We consider J_m and apply the local model at J_m.
- It shows that only stable orbits in the vicinity of J_m correspond to diagonalizable matrices with *distinct* eigenvalues!
- In fact, there is a remarkable *slice* at J_m which parametrizes all orbits whose *minimal* polynomials are of degree m.
  This slice contains classical *companion forms* of these orbits.

Spectrum block-partition

Let y be in Jordan canonical form with s eigenvalues $\mu_1, \ldots, \mu_s$.
For μ_i, we define the block-partition $\lambda_i = (\lambda_{i1} \geq \lambda_{i2} \geq \ldots)$ which records the sizes of Jordan blocks of type μ_i
The spectrum block-partition λ of y is the sum $\lambda = \lambda_1 + \ldots + \lambda_s$

$$y = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

The block-partitions are $(2, 1, 1)$, (2) for eigenvalues -1 and 3
The spectrum block-partition of y is $(2, 1, 1) + (2) = (4, 1, 1)$

If y has m distinct eigenvalues, its spectrum block-partition is (m).

Projective orbit closure theorem

Main Theorem

Let x be a nilpotent matrix whose nilpotent block-partition is θ. and y be a matrix whose spectrum block-partition is λ
The projective orbit closure of y contains x iff $\theta \trianglelefteq \lambda$.

The proof uses the following G-stable projective varieties.

$$\mathcal{X}_k^r = \left\{ z \;\middle|\; \begin{array}{c} \exists \text{ eigenvalues } \mu_1, \mu_2, \ldots, \mu_k \text{ of } z \text{ such that} \\ \mathrm{rank}((z - \mu_1 I) \cdots (z - \mu_k I)) \leq r \end{array} \right\}$$

- If $\theta \ntrianglelefteq \lambda$, we show there exists r and k such that $y \in \mathcal{X}_k^r$ and $x \notin \mathcal{X}_k^r$
- Otherwise, we exhibit a 1-PS such that the projective limit of y under this 1-PS is a nilpotent matrix with block-partition λ.

Illustration of the theorem

Consider the following matrices in Jordan canonical forms

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad y' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- The spectrum block-partition of y is $(1,1)+(1)=(2,1)$
- The spectrum block-partition of y' is $(2)+(1)=(3)$.
- The nilpotent block-partition of x is (3)

We conclude that

- $y \in \overline{O(y')}$ (classical)
- $[x] \notin \overline{O([y])}$ (our theorem)
- $[x] \in \overline{O([y'])}$ (our theorem)

Local model -an introduction

Objective

Build a Luna-style model of a neighborhood of an arbitrary point p.
H - stabilizer subgroup of p and $\mathcal{H}$ - its stabilizer Lie subalgebra.
H, $\mathcal{H}$ may not be reductive

RECALL Luna slice theorem for a stable point p

$O(p)$ - closed, $H = G_p$ reductive and N - a normal H-complement

- Commuting actions of H and G on $G \times N$

$$h \cdot (g, n) = (gh^{-1}, h.n) \qquad g' \cdot (g, n) = (g'g, n)$$

  The H-quotient is $G \times_H N$; G acts on $G \times_H N$

- $G \times_H N$ is the normal bundle of the orbit $O(p) = G/H$ in V
- The map $\mu : G \times N \to V$ defined as $\mu(g, n) = g \cdot (p + n)$ is G-equivariant and constant on H-orbits

$$\mu(gh^{-1}, h.n) = gh^{-1}(p + h.n) = g.(p + n)$$

  So, μ descends to a G-equivariant map $\phi : G \times_H N \to V$.

$$\phi([(g, n)]_H) = g.(p + n)$$

Local model: p is arbitrary

$O = O(p)$ - ~~closed~~, $H = G_p$ ~~reductive~~ ~~and N - a normal H-complement~~ *We initially work with V instead of N*

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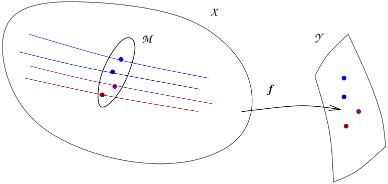
The quotient construction for Lie algebra actions

So far, we have G-equivariant $\phi : G \times_H V \to V$.

We consider induced Lie algebra $\mathcal{G}$-actions on $G \times_H V$ and V; $\phi : G \times_H V \to V$ is also $\mathcal{G}$-equivariant.

Lie algebra quotient action

- Let $\mathcal{X}_m$ and $\mathcal{Y}_n$ be two manifolds with $m \geq n$, and $\mathcal{G}$-actions, $\rho : \mathcal{G} \to Vec(\mathcal{X})$ and $\eta : \mathcal{G} \to Vec(\mathcal{Y})$.
- Let $f : \mathcal{X} \to \mathcal{Y}$ be *regular* at $x \in \mathcal{X}$, with $f(x) = y$ such that f is $\mathcal{G}$-equivariant. For $x \in \mathcal{X}$, let $\mathcal{D}(x) = ker(f_x^*)$. $\mathcal{D}$ is a distribution of rank $(m - n)$.
- Let $\mathcal{M}_{m-n}$ be a submanifold of $\mathcal{X}$, transverse to the kernel distribution $\mathcal{D}$. Then for any $m \in \mathcal{M}$ and $\mathfrak{g} \in \mathcal{G}$, define $\rho_{\mathcal{M}}(\mathfrak{g})(m)$ as $\pi_{\mathcal{M}}(\rho(\mathfrak{g})(m))$, the projection onto the tangent space $T\mathcal{M}_m$. Then $\rho_{\mathcal{M}}$ is a $\mathcal{G}$-action on $\mathcal{M}$



Let $\mathcal{M}_{m-n}$ be a submanifold of $\mathcal{X}$, transverse to the kernel distribution $\mathcal{D}$. Then for any $m \in \mathcal{M}$ and $\mathfrak{g} \in \mathcal{G}$, define

$$\rho_{\mathcal{M}}(\mathfrak{g})(m) = \pi_{\mathcal{M}}(\rho(\mathfrak{g})(m))$$

where $\pi_{\mathcal{M}}(\rho(\mathfrak{g})(m))$ is the projection from $T\mathcal{X}_m$ onto the tangent space $T\mathcal{M}_m$.
Note that $T\mathcal{X}_m = T\mathcal{M}_m \oplus kernel(f_m^*)$
Then $\rho_{\mathcal{M}}$ is a $\mathcal{G}$-action on $\mathcal{M}$

The local model

We want to apply the quotient construction to $\phi : G \times_H V \to V$
We choose a submanifold which is transverse to the kernel of ϕ at the point $[e, 0]_H$

Key choices

Let $M \subset G$ - a submanifold containing $e \in G$ and complementary/transversal to H at e; $\mathcal{G} = \mathcal{H} \oplus T_e M$
Let N be a vector subspace complement to $T_p O$ in V. This slice N may not be H-invariant.

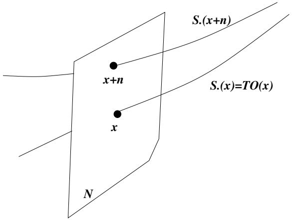
The Local Model

- Fix choices M and N

$$\mu : M \times N \to V : \mu(m, n) = m \cdot (x + n)$$

- $M \cdot x$ is the orbit, $M \cdot (x + n)$ is merely the same dimension of $M \cdot x$ and transversal to N.
- We simply analyze its derivative at $(e, 0)$
  - derivative of μ^* along M maps to the orbit tangent space.
  - derivative of μ^* along N maps to the slice N

This shows that μ is a local diffeomorphism.



Embedding the local model inside $G \times_H V$

- The map $\mu : G \times V \to V$ given by $\mu(g, v) = g \cdot (x + v)$ factorizes through $G \times_H V$.

$$\mu(gh, h^{-1}v) = gh(x + h^{-1}V) = g(x + v) \text{ since } h \cdot x = x$$

$$\begin{array}{ccccc} G \times V & \rightarrow & G \times_H V & \rightarrow & V \\ & & \uparrow & \nearrow & \\ M \times N & \rightarrow & M \times V & & \end{array}$$

- Thus, $\phi : G \times_H V \to V$ is $\mathcal{G}$-equivariant and regular at $(e, 0)$.
- $M \times N$ is transversal to the kernel of this map.
- This $\mathcal{G}$-space $M \times N$ is the local model.

This shows existence and construction of the local model. We need explicit $\mathcal{G}$ action on $M \times N$. To be described in the main talk.

Summary

Local model

- We develop a Luna-type model which applies to all points.
- It is local - we restrict to a nbd of the point and not the entire orbit, and work with Lie algebra action in this nbd.
- We provide an explicit description of the Lie algebra action on the model. (next talk)

Application of local model

We apply the local model to the conjugation action to understand projective orbit closures.

Preview of next talk by Milind Sohoni

- The Big Picture
- The Local Model
  - Quotients of lie algebra action on $G \times_H V$.
  - The local model expressions and its properties.
- Forms - Limits and the local stabilizers
  - The family $f(t) = A(t) \cdot f = g + t^b f_b + \ldots$, with stabilizers $\mathcal{H}$ and $\mathcal{K}$ of g and f.
  - A basis for $\mathcal{K}(t)$ and its properties - $\mathcal{K}_0$ and $\overline{f_b}$.
  - The special case of $\lambda(t)$ and the triple-stabilizer conditions.
  - The co-dimension-1 case.
- Advanced topics

Thank You!